BOIJ-SÖDERBERG EXPANSIONS OF MATROID STANLEY-REISNER RINGS

ALEX FINK

This note records a proof of Proposition 0.1 below, on a decomposition of matroid Stanley-Reisner rings into pure Boij-Söderberg tables. We take the fundamental pure tables to be the vectors $\pi_{\mathbf{d}} \in \mathbb{Q}^{\mathbb{Z}^2}$ indexed by sequences of positive integers $\mathbf{d} = (d_0, \ldots, d_c)$, such that the only nonzero components of $\pi_{\mathbf{d}}$ are

$$(\pi_{\mathbf{d}})_{id_i} = \frac{(-1)^i}{\prod_{j \neq i}(d_j - d_i)}.$$

We will always have $d_0 = 0$. We also write $\{e_{ij}\}$ for the standard basis for the space $\mathbb{Q}^{\mathbb{Z}^2}$ of Betti tables.

Let $S = \mathbf{k}[x_1, \ldots, x_n]$. If Δ is a simplicial complex on $[n] = \{1, \ldots, n\}$, then $I_\Delta \subseteq S$ will denote its Stanley-Reisner ideal. Matroids on the ground set $[n]$ are interpreted as certain simplicial complexes on the vertices $[n]$, whose faces are the independent sets: thus the rank of M is its dimension plus one. We use matroidal notation for operations on these complexes: for instance we denote restriction of the complex Δ to a set A by $\Delta|A$.

For concision, let $\mathcal{C}(M)$ be the set of maximal chains of flats of a matroid M. If the ground set of M is $[n]$, this is the set of tuples $\mathbf{F} = (F_0, \ldots, F_{\operatorname{rk} M})$ in which

$$\emptyset = F_0 \subsetneq \cdots \subsetneq F_{\operatorname{rk} M} = [n]$$

are all flats.

Proposition 0.1. *If M is a matroid on $[n]$ of rank r with no coloops, then the Betti table of the Stanley-Reisner ring S/I_M is given by*

$$\beta(S/I_M) = \sum_{\mathbf{F} \in \mathcal{C}(M)} \left(\prod_{i=1}^{n-r} |F_i| - |F_{i-1}| \right) \cdot \pi_{n-|F_{n-r}|, \ldots, n-|F_0|} \tag{0.1}$$

Proof. We will use Hochster's formula [1, Corollary 5.12], in the following form:

$$\beta_{ij}(S/I_M) = \sum_{\substack{A \subseteq [n] \\ |A| = j}} \dim \widetilde{H}^{j-i-1}(M|A, \mathbf{k}).$$

These restrictions $M|A$ of the matroid M are themselves matroids and are therefore Cohen-Macaulay, and so $\dim \widetilde{H}^{j-i-1}(M|A, \mathbf{k})$ is only nonzero if $j-i-1$ is equal to the dimension of $M|A$, i.e. if $j - i = \operatorname{rk}_M(A)$. The dimension of the top-dimensional homology of $M|A$ is the Tutte evaluation $T_{M|A}(0, 1)$. So the above sum may be recast

$$\beta(S/I_M) = \sum_{A \subseteq [n]} T_{M|A}(0, 1)\, e_{|A| - \operatorname{rk}_M(A), |A|}.$$

Changing to the dual matroid, and writing $F = [n] \setminus A$, this is

$$\beta(S/I_M) = \sum_{F \subseteq [n]} T_{M*/F}(1,0)\, e_{n-r-\mathrm{rk}_{M^*}(F), n-|F|}. \tag{0.2}$$

Let us now turn to the right side of (0.1). Expanding the definition of the $\pi_{\mathbf{d}}$, this is

$$\sum_{\mathbf{F}} \sum_{i=0}^{n-r} e_{n-r-i, n-|F_i|} (-1)^{n-r-1} \frac{\prod_{j=1}^{n-r} |F_j| - |F_{j-1}|}{\prod_{j \neq i} |F_j| - |F_i|}.$$

We recast this as a sum over the various flats $F := F_i$ of M^* that occur in the chains $\mathbf{F}$, breaking up the remaining summation into the subchain of $\mathbf{F}$ before the ith position and the subchain after. Note that $i = \mathrm{rk}_{M^*}(F)$. What results is

$$\sum_{F \text{ a flat}} e_{n-r-\mathrm{rk}_{M^*}(F), n-|F|} \left(\sum_{\mathbf{G} \in \mathcal{C}(M^*|F)} \prod_{j=1}^{\mathrm{rk}\, M^*|F} \frac{|G_j| - |G_{j-1}|}{|F| - |G_{j-1}|} \right) \left(\sum_{\mathbf{H} \in \mathcal{C}(M^*/F)} \prod_{j=1}^{\mathrm{rk}\, M^*/F} \frac{|H_j| - |H_{j-1}|}{|H_j|} \right).$$

We now compare this sum to (0.2). First of all, the terms of (0.2) for which F is not a flat of M^* make no contribution, as then M^*/F contains a loop, making $T_{M*/F}(1,0)$ equal to 0. We are thus done in view of the equations in Lemma 0.2 for the two parenthesized factors. ($M^*|F$ is loopfree because M^* is; M^*/F is because F is a flat.) □

Lemma 0.2. *Let M be a matroid on ground set $[n]$ with no loops. Then*

(a) $\displaystyle\sum_{\mathbf{F} \in \mathcal{C}(M)} \prod_{j=1}^{\mathrm{rk}\, M} \frac{|F_j| - |F_{j-1}|}{n - |F_{j-1}|} = 1.$

(b) $\displaystyle\sum_{\mathbf{F} \in \mathcal{C}(M)} \prod_{j=1}^{\mathrm{rk}\, M} \frac{|F_j| - |F_{j-1}|}{|F_j|} = T_M(1,0).$

Proof. In both cases the proof will be inductive on the rank of M, by taking subchains of length one less and passing to an appropriate minor of M. The rank 0 base cases are trivial.

For (a), we extract the $j = 1$ term of the product, giving

$$\sum_{F \text{ a rank 1 flat}} \left(\frac{|F|}{n} \sum_{\mathbf{G} \in \mathcal{C}(M/F)} \prod_{i=1}^{\mathrm{rk}\, M - 1} \frac{|G_i| - |G_{i-1}|}{(n - |F|) - |G_{i-1}|} \right)$$
$$= \sum_{F \text{ a rank 1 flat}} \frac{|F|}{n} \cdot 1$$

by induction. Since the rank 1 flats partition $[n]$, the sum above equals 1 as desired.

For (b), we begin by noting that $T_M(1,0)$ is the Möbius function evaluation $(-1)^{\mathrm{rk}\, M} \mu(\emptyset, [n])$ in the lattice of flats of M. (This follows from the Crosscut Theorem [2, Corollary 3.9.4], since by the corank-nullity expansion of Tutte, $T_M(1,0)$ counts spanning sets of M with alternating sign.)

Using the induction, we extract the $j = \operatorname{rk} M$ term of the product and have

$$\begin{aligned}
&\sum_{F \text{ a hyperplane}} \left(\frac{n - |F|}{n} \sum_{\mathbf{G} \in \mathcal{C}(M|F)} \prod_{j=1}^{\operatorname{rk} M - 1} \frac{|G_j| - |G_{j-1}|}{|G_j|} \right) \\
&= \sum_{F \text{ a hyperplane}} \frac{n - |F|}{n} \cdot (-1)^{\operatorname{rk} M - 1} \mu(\emptyset, F) \\
&= \frac{1}{n} \sum_{a \in [n]} \sum_{F \not\ni a \text{ a hyperplane}} (-1)^{\operatorname{rk} M - 1} \mu(\emptyset, F) \\
&= \frac{1}{n} \sum_{a \in [n]} (-1)^{\operatorname{rk} M} \mu(\emptyset, [n]) \\
&= (-1)^{\operatorname{rk} M} \mu(\emptyset, [n]),
\end{aligned}$$

where the second-last equality is Weisner's theorem [2, Corollary 3.9.3]. □

[[Eliminate the no-coloops restriction. Is this better framed in terms of the cover ideal, and does it then go through for non-matroids? Are there connections between the product on Boij-Söderberg tables and my Hopf structures with Derksen?]]

References

[1] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*.
[2] Richard Stanley, *Enumerative combinatorics* vol. 1, 2nd ed.