Lattice games and computation

Alex Fink

North Carolina State University

Games At Dal
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Representing taking-and-breaking games

What's a good way to represent positions of taking-and-breaking games? E.g. the Kayles position .

- As the tuple $(5, 2)$?

  A move is reducing a component i to $j < i$ and introducing a new component $i - j - 1$ or $i - j - 2$.

- The approach used in misère quotient theory: [Plambeck-Siegel] consider each possible disjunctive summand (heap), one at a time, and completely understand what happens when adding it to known positions.

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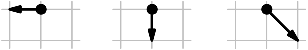
The positions of a heap game with allowable heaps $1, \ldots, n$ make up $\mathbb{N}^n$:

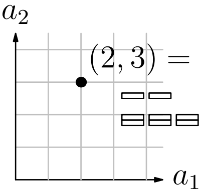
$$(a_1, \ldots, a_n) = a_1 \text{ 1s}, \ \ldots, \ a_n \ n\text{s}$$

Valid moves correspond to subtracting a vector chosen from a fixed finite set.

Example: Nim on heaps of ≤ 2. Valid moves:

reduce a heap of	1 to 0	2 to 0	2 to 1
i.e. subtract	$(1, 0)$	$(0, 1)$	$(-1, 1)$

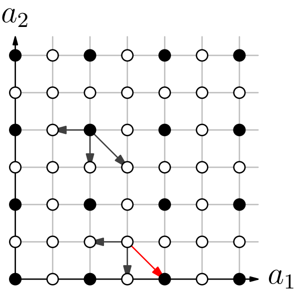




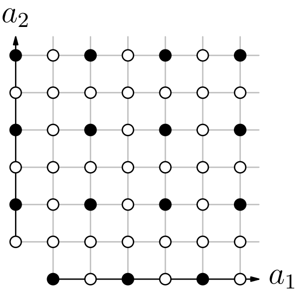
This is an example of a lattice game.

Legend: $\bullet$ = P-position; $\circ$ = N-position.

Normal play



Misère play



A lattice game is an impartial game whose positions are a subset $\mathcal{B} \subseteq \mathbb{Z}^n$, a f. g. module for a pointed normal f. g. affine semigroup

(main examples: $\mathcal{B} = \mathbb{N}^n$ possibly with a bite out of the corner)

where the options of x are $\{x - \gamma\}$ for $\gamma \in \Gamma$, the ruleset. [Guo-Miller]

Aim: apply monoid theory, polyhedral geometry, commutative algebra. . .

Example 1: Heap games.
Example 2: 1-D lattice games are subtraction games (in the ~~first representation~~).

Other technicalities:

- the game should always end. . . $\Longrightarrow$ Γ generates a pointed cone
- near the generators of $\mathcal{B}$. $\Longrightarrow$ "tangent cone axiom"

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What do we want from a strategy?

Strategies for determining outcome class, and good moves, should be efficient: polynomial time in the input size, $\sim \log(\#\text{ heaps})$.

We take the heap sizes to be less than a universal constant n.

In normal play, Sprague-Grundy says heap games have efficient strategies:

store the G-values $\mathcal{G}(1), \mathcal{G}(2), \ldots, \mathcal{G}(n)$; then

$$a_1 \text{ 1s}, \ \ldots, \ a_n \ n\text{s is } \mathrm{P} \qquad \Longleftrightarrow \qquad \bigoplus (a_i \bmod 2) \cdot \mathcal{G}(i) = 0.$$

In misère play, the Plambeck-Siegel misère quotients provide efficient strategies if they are finite.
But they might not be, even with bounded heap size.

A lattice game on a $\mathbb{N}^n$-module is squarefree if each move decreases just one coordinate, by just one. [GM erratum]

Example: heap games only destroy one heap in a move.

Prop'n. (GM) A lattice game on $\mathbb{N}^n$ (i.e. normal play) is squarefree $\iff x + y$ is the disjunctive sum of x and y.

Let $\mathcal{P}$ be the set of P-positions. Sprague-Grundy says:

Thm. (GM) In a squarefree lattice game in normal play,

$$\mathcal{P} = (2\mathbb{N})^n + (\mathcal{P} \cap \{0,1\}^n)$$

Squarefree lattice games: the easiest lattice games

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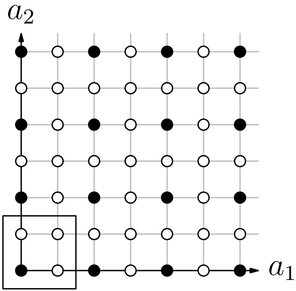
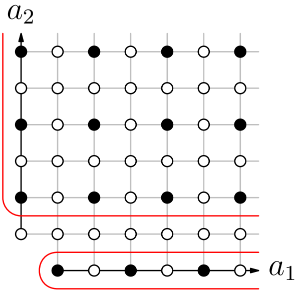


$\mathcal{P} = (2\mathbb{N})^2 + \{(0,0)\}$

gen. func. $\dfrac{1}{(1-t_1^2)(1-t_2^2)}$

Misère play



$$\mathcal{P} = ((2\mathbb{N})^2 + \{(0,2)\}) \ \dot\cup\ \mathbb{N}(2,0) + \{(1,0)\}$$

$$\frac{t_2^2}{(1-t_1^2)(1-t_2^2)} + \frac{t_1}{1-t_1^2}$$

An affine stratification for a lattice game is a way to decompose its P-positions into a finite number of purely periodic polyhedral regions.

…that is, a finite union $\bigcup(\text{polyhedron} \cap \text{sublattice})$.

Equivalently: the P-positions have a rational generating function (rational strategy).

Example: normal play squarefree games.

Theorem. (GM) An affine stratification gives an efficient strategy.

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Main theorem

Conjecture. (GM) Every lattice game has an affine stratification ($\Longrightarrow$ an efficient strategy).

Nope.

Theorem. (—) Lattice games on $\mathbb{N}^3$-modules are computationally universal.

In particular, given $M, N, a, b \in \mathbb{N}$, questions like

> *Does a given lattice game have any* P*-positions of form* $(Mi + a, Nj + b, 1)$*?*

can be undecidable.

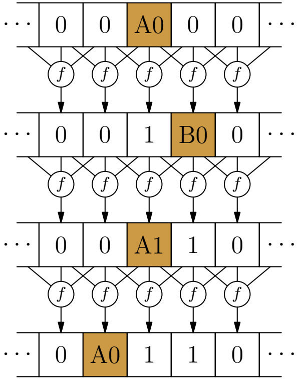
This is even true if Γ is fixed, or if $\mathcal{B} = \mathbb{N}^3$.

Proof strategy: implement a Turing machine as a lattice game.

Let $\mathrm{P} = \textit{true}$ and $\mathrm{N} = \textit{false}$.
A position's outcome is the NOR of its options' outcomes.

Any boolean function can be constructed as a circuit of NORs.

If T is a Turing machine, the behaviour of T can be computed by a doubly periodic NOR circuit.

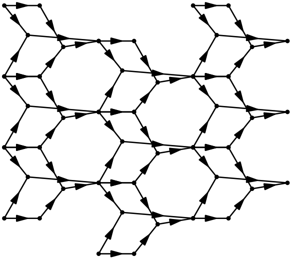


An engineering problem

How to implement an arbitrary NOR circuit with a single ruleset?

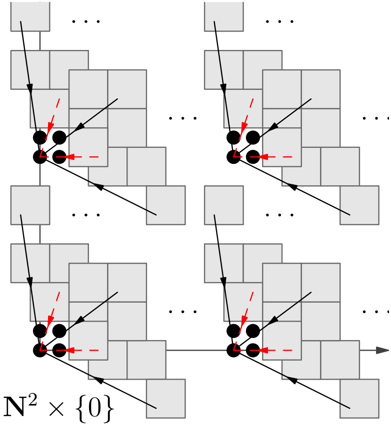
Not hard if you could declare positions illegal. Put the gates in generic positions in $\mathbb{N}^2$ and make everything else illegal.

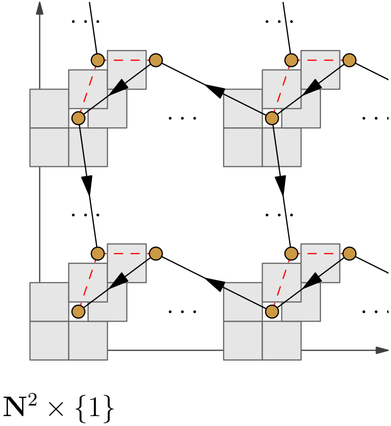
Then each ruleset element dictates the presence or absence of at most one wire in the circuit.



An engineering problem (2)

Actually, we force the "illegal" positions to be N-positions in $\mathbb{N}^2 \times \{1\}$, by providing moves down to P-positions in $\mathbb{N}^2 \times \{0\}$:

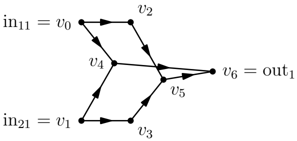




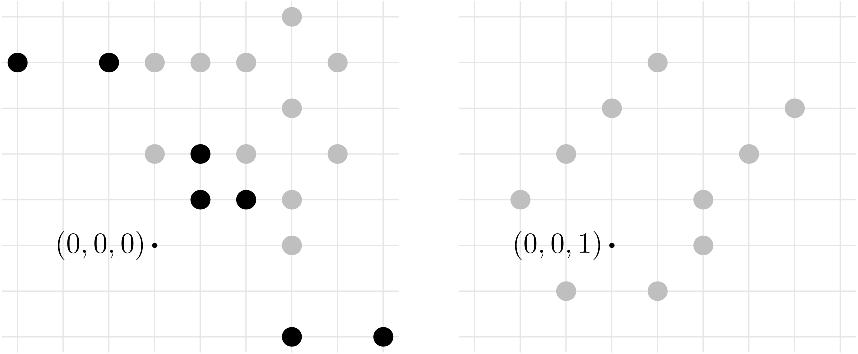
An explicit lattice game with no affine stratification

Let's build the Sierpiński gasket.

$$f(i,j) = f(i-1,j) \text{ XOR } f(i,j-1)$$



The ruleset:



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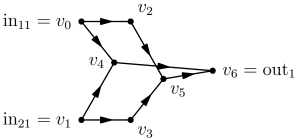
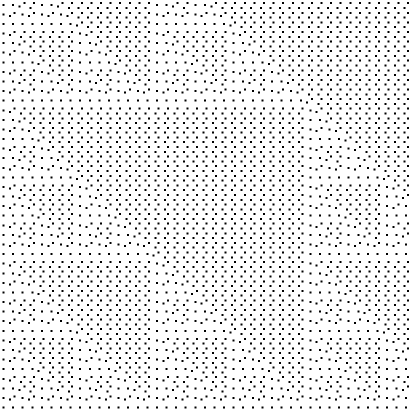
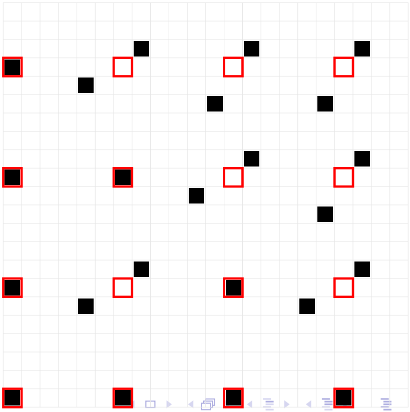


The P-positions of form $(x, y, 1)$:





Where next?

Ezra Miller's latest:

Theorem. A lattice game with finite misère quotient has an affine stratification.

Question: Where does the border of the efficient strategies lie within lattice games?

E.g. squarefree games in misère (or more general) play?

Thanks!

- A. Guo and E. Miller, *Lattice point methods in combinatorial games*, Adv. in Appl. Math. **46** (2011), 363–378.
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