The Number-Pad Game

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Problem 3 in the 1st Mathematical Olympiad of Central America and the Carribean, held in Costa Rica in 1999 [2] was the following:

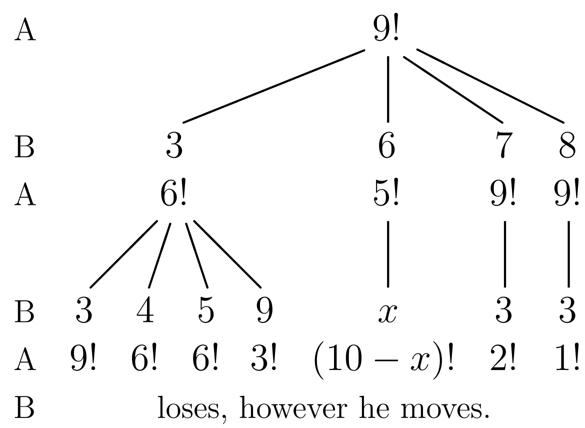
> In a calculator the number keys (except 0) are arranged as shown. Player A turns on the calculator, presses a digit key and then presses the + key. Then a second player B presses a digit key in the same row or column of the last digit key pressed by A, except the same key pressed by A, then presses the + key. The game proceeds with the two players taking turns alternately. The first player who reaches a sum greater than 30 loses. Which player has a winning strategy? Describe the strategy.

7	8	9
4	5	6
1	2	3

Number pads for telephones are usually the opposite way up from those on computers, but it doesn't make any difference to the game, in case you're used to the different arrangement.

You can visualize the game as being played by stacking wooden blocks on top of one another: when the height exceeds 30 blocks, the stack topples and the player who was responsible loses.

We analyze this game, first with a **tallest tolerable tower** of height $t = 30$, and then for a general value of t. For $t = 30$, A can win by touching 9. Here is a complete strategy, with an exclamation mark (!) indicating A's winning moves:



A can also win by touching 3, but B, by touching 1 at each turn, can prolong the game for five more moves by A:

$$\text{A3 B1 A7 B1} \begin{cases} \text{A4 B1 A4 B1} \begin{cases} \text{A4 B1 A3} \\ \text{A3 B1 A4} \end{cases} \\ \text{A2 B1} \begin{cases} \text{A4 B1 A7 B1 A2} \\ \text{A2 B1 A7 B1 A4} \end{cases} \end{cases}$$

Table 1 tells you how to play well for any tallest tolerable tower. It shows all the winning moves. Each entry was found by reference to earlier entries. For example, if $t = 15$, then

if you touch	leading to	where the winners	so that
1	$t = 14$	9 can't be reached	**1 wins**
2	13	4,9 can't be reached	**2 wins**
3	12	1 can be reached	3 loses
4	11	8,9 can't be reached	**4 wins**
5	10	1,5,7,9 can't be reached	**5 wins**
6	9	9 can be reached	6 loses
7	8	5 can be reached	7 loses
8	7	4 can be reached	8 loses
9	6	3 or 5 can be reached	9 loses

and so the entry for $t = 15$ is 1245.

t	0	1	2	3	4	5	6	7	8	9
0+		1	12	3	4	5	12356	3467	34568	89
10+	1579	89	178	49	9	1245	2	4	246	25
20+	234	4	69	1279	2349	489	79	—	1249	248
30+	39	45	5	146	347	358	9	1359	23	7
40+	49	2589	2345	—	145	2679	37	24	5	12369
50+	7	3458	29	3579	23	57	469	79	2345	2
60+	5	1279	57	34	—	1356	237	3457	49	579
70+	236	78	48	79	35	23	4	3679	25	345
80+	48	7	37	23	348	379	5	134569	478	37
90+	57	35	38	37	35	359	3457	37	57	35
100+	3578	37	35	35	357	37	357	35	357	357
110+	357	357	357	357	357	357	357	357	...	

Table 1. Winning moves for $0 < t \leq 117$.

Notice that there are no winning moves if the tallest tolerable tower has height 27 or 43 or 64. For example, at $t = 27$,

the moves	1	2	3	4	5	6	7	8	9
leading to $t =$	26	25	24	23	22	21	20	19	18
are met by	7	8	9 or 2	7, 5 or 1	6	4	4	5 or 2	6

respectively. These are $\mathcal{P}$**-positions**, previous-player-winning positions [1, Vol.1, p.83], where A had better hope that it is B who has to start.

Here's how to use the table. Suppose that the tallest tower has height 100. From the table, the winning moves are 3, 5, 7, and 8, so, to win as quickly as possible, touch 8. Your opponent may now move to 2, 5, 7, or 9, and the height of the tallest tower that can be added is accordingly reduced to 90, 87, 85, or 83. From the table, your good moves are then respectively 5 (7 is not legal); 4 or 6 (1, 3, 5, 9 are illegal); 9 (3 and 7 aren't legal); or 3 (2 is illegal). You will find that there is always a legal winning move, unless the game starts at 27, 43, or 64. For towers taller than 107, you can always win by playing 3, 5, or 7 — your opponent can never reply with one of these numbers, and whatever he plays, you can reply with either of two of them. When the remaining allowable height is 100 or less, you have to be more careful and use the table.

As described here, the game is being played as a **misère** game, with the last player losing. We could also play it under **normal play** rules, in which the last player wins, that is, the winner is the first to exceed the tallest tolerable total and topple the tower.

This is an example of a subtraction game played as an addition game. It is also a subtraction game in which the subtraction set depends on the previous move. As a subtraction game it would be more natural to disallow the tower-toppling move, which would switch the application of the labels “misère” and “normal”.

It is not hard to see that subtraction games are ultimately periodic, but sometimes the periodicity takes a long time to settle down. In the misère game, for example, the period has length 1 (repetition of the single entry 357), after a “preperiod” of length 107. Often there seems to be no good way of estimating the lengths of the period and preperiod. Table 2 shows the winning moves for the normal-play game.

t	0	1	2	3	4	5	6	7	8	9
0+	any	23...9	34...9	45...9	56789	6789	789	89	19	1
10+	1	14	—	15	125	356	46	589	125689	789
20+	89	19	179	9	148	2	5	1245	5	134
30+	9	1356	2789	57	49	579	239	89	145	9
40+	1457	24	—	1249	25	34	4	59	12369	347
50+	489	79	359	289	48	9	159	12345	4	4
60+	25	235	348	4	9	1239	2348	4789	479	59
70+	1248	8	19	19	135	4	—	1256	2359	3458
80+	4	59	12369	378	489	9	135	2	48	249
90+	59	12345	4	46	279	235	2489	—	19	**129**
100+	348	**489**	57	356	47	**348**	**29**	359	**2389**	47
110+	**47**	2	**2345**	**238**	—	*149*	2	**346**	247	5
120+	12469	7	368	*279*	**359**	*289*	*57*	6	179	*2345*
130+	8	15	1	*13567*	47	8	1256	35	*368*	4
140+	389	269	35	248	47	3	*28*	3458	*39*	47
150+	345	2	3467	*2358*	—	*14*	2	36	247	5
160+	12469	7	368	*2789*	3579	*89*	*157*	6	179	*245*
170+	8	15	1	*1356*	47	8	1256	35	*3689*	4
180+	389	269	35	248	47	3	*2*	3458	*239*	47
190+	345	2	3467	*2378*	—	*149*	2	36	247	5
200+	12469	7	368	*279*	3579	*289*	*57*	6	179	*2345*
210+	8	15	1	*13567*	47	8	1256	35	*368*	4

Table 2. Winning moves for normal play of the number-pad game.

Notice that 12, 42, 76, 97, and $40k+114$ for $k \geq 0$ are $\mathcal{P}$-positions. The period length is 80, and the preperiod is 124. The last **irregular set** is **359** for $t = 124$, shown in **bold** in the table, as are the other last irregular sets in each column. The period would be 40, were it not for a quarter of the sets, which are *italicized* in the table.

We next consider the same game but with a zero key in the middle column of a separate row. In the misère game, in which you're trying to avoid toppling, 0 will always be a good move since it never topples, and, if the reply (which must be 2, 5, or 8) doesn't immediately lose, then 0 is repeatedly available until it becomes the penultimate move. So it is forbidden as an opening move. Players soon realize that 2, 5, and 8 can never be good moves, since 0 is always a good reply, so the analysis can be simplified to the board on the right of the following diagram.

7	8	9
4	5	6
1	2	3
	0	

7	9
4	6
1	3

The winning moves are shown in Table 3: the length of the period is 15 and that of the preperiod is 22. The last of just eight irregular sets (shown in **bold**) is **1679** for $t = 22$. The $\mathcal{P}$-positions are those with $t \equiv 0$ or $5 \pmod{15}$.

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0+	—	1	**1**	**13**	**4**	—	16	**1367**	**346**	49	9	169	**167**	49	49
15+	—	1	14	**3469**	49	—	16	**1679**	349	49	9	169	1467	49	49
30+	—	1	14	349	49	—	16	14679	349	49	9	169	1467	49	49
45+	—	1	14	349	49	—	16	14679	349	49	9	169	1467	49	49

Table 3. Winning moves in the number-pad game with zero.

In the normal play of this game, 0 is *never* a good move, so the analysis is the same as that for normal play in the game *without* zero.

Sometimes the zero key is in the left hand column instead of being in the middle. If so, in the misère game, 1, 4, and 7 will be bad moves. The winning moves will be as in Table 4. The length of the period is now 11 and the last of the irregular sets, which are again shown in **bold**, is **389** for $t = 36$. The $\mathcal{P}$-positions are 0, 1, 12, and $11k + 28$ and $11k + 29$ for $k \geq 0$.

t	0	1	2	3	4	5	6	7	8	9	10
0+	—	—	**2**	**23**	**23**	**5**	**56**	**6**	**368**	**289**	**59**
11+	**9**	—	**26**	**29**	**2358**	**258**	**5**	**6**	**69**	**2389**	**29**
22+	5	**29**	6	**23**	289	**5**	—	—	**2369**	239	**3589**
33+	5	56	6	**389**	289	59	—	—	26	239	2358
44+	5	56	6	3689	289	59	—	—	26	239	2358
55+	5	56	6	3689	289	59	—	—	26	239	2358

Table 4. Number-pad winners if zero is on the left.

Or, perhaps the zero key on your computer covers both the first two columns, making all of 1, 2, 4, 5, 7, 8 into bad moves! This is a good game to play with your friends, once you've mastered it yourself. Hint: if it's noon or one or two o'clock, invite your opponent to move. Otherwise touch 3, unless it's nine, ten, or eleven o'clock, in which case touch 9.

There are many variations on subtraction games, some of which are far from being fully understood. See the Unsolved Combinatorial Games section and the Bibliography section of the forthcoming book [3].

References

[1] Elwyn Berlekamp, John Conway & Richard Guy, *Winning Ways for your Mathematical Plays*, AKPeters, 2003.

[2] José H. Nieto Said & Rafael Sánchez Lamoneda, The Mathematical Olympiad of Central America and the Caribbean, *Mathematics Competitions*, **18**:1(2005) 16–38. Journal of the World Federation of National Mathematics Competitions, AMT Publishing, University of Canberra ACT 2601 Australia.

[3] Richard Nowakowski (editor), *Games of No Chance III*, Cambridge University Press, 2007.