Patulous Pegboard Polygons

Derek Kisman, Richard Guy & Alex Fink

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A problem in a recent competition was:

Given a 2004 by 2004 square grid of dots, what is the largest number of edges of a convex polygon whose vertices are dots in the grid?

Of course, the question can be asked for any value of 2004, say n. For $n = 2$, 3 and 4 it's easy to see (Figure 1) that the answers are $p = 2n$:

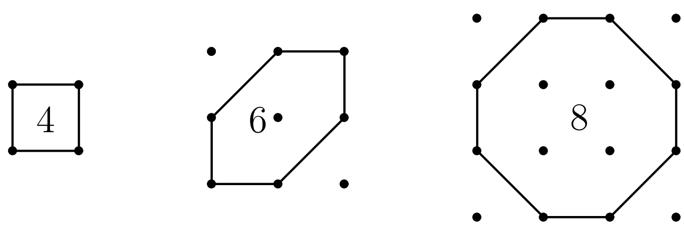

Figure 1: The best polygons for $n = 2$, 3 and 4

The next three (Figure 2) are not quite so obvious:

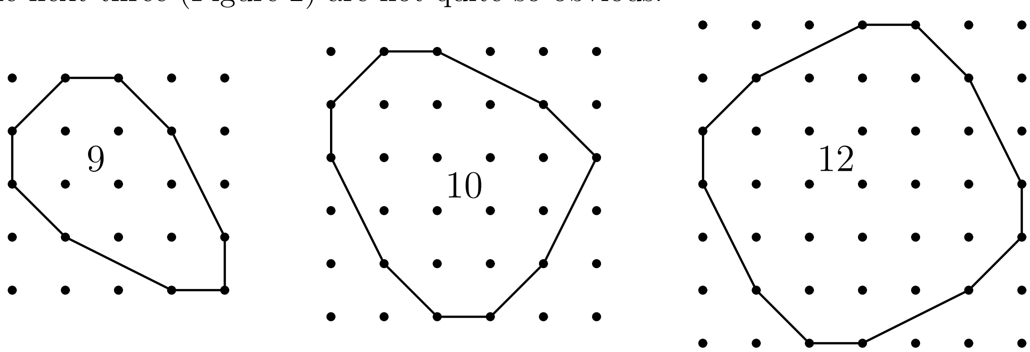

Figure 2: The best polygons for $n = 5$, 6 and 7

How can we be sure that they are the best? As we go round the polygon, which we will always do counterclockwise, each edge has a certain **cost**, namely the sum of its N-S and E-W components. The total cost must not exceed $4(n-1)$, i.e., at most $n-1$ in each of the four cardinal directions. There are just four available edges of cost 1, and four of cost 2. The former are used in all our polygons so far, except that the northbound one is missing when $n = 6$. Two of the latter are used when $n = 3$ and all of them subsequently, except for the northeasterly one when $n = 5$. Edges of cost 3 are like knight's moves in chess and occur 2, 3 and 4 times in our polygons for $n = 5$, 6 and 7, respectively. It's not too hard to see that whenever the answer is a multiple of 4 (as it is when $n = 2$, 4 or 7) we get it by using the cheapest possible edges. Edges have components a, b, say, so that their cost is $a + b = c$, which we keep to a minimum by assuming that $a \perp b$, that is, that a and b have no common factor bigger than 1. This is safe in the sense that if any polygon fits into a square grid then one with reduced sides will. We have seen that, when $c = 1$ or 2, there are just 4 different edge-directions. When $c \geq 3$, $a \neq b$ and there are just 4 orthogonal directions, $(\pm a, \pm b)$ and $(\pm b, \mp a)$, for each ordered pair (a, b).

The number of such pairs (a, b) for edges of cost $c = a + b$ is $\phi(c)$, Euler's totient function, the number of numbers from 1 to c which are prime to c. The first few values are the second row in the following table, which we have extended far enough to answer the opening question.

$c =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$\phi(c)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8	8	16	6	18	8	12
$c\phi(c)$	1	2	6	8	20	12	42	32	54	40	110	48	156	84	120	128	272	108	342	160	252
$\sum c\phi(c)$	1	3	9	17	37	49	91	123	177	217	327	375	531	615	735	863	1135	1243	1585	1745	1997
n	2	4	10	18	38	50	92	124	178	218	328	376	532	616	736	864	1136	1244	1586	1746	1998
$p = 4\sum \phi(c)$	4	8	16	24	40	48	72	88	112	128	168	184	232	256	288	320	384	408	480	512	560

The third row is a quarter of the total cost of all edges of cost c, and the fourth row is the cumulative total, whose values are 1 less than values of n for which there is an optimal solution, shown in the fifth row.

Since we can introduce new edges in orthogonal sets of four, we can also include the values of n which, for each k, allow an optimal polygon with $p(n) = 4k$ edges:

$k=$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$n=$	2	4	7	10	14	18	23	28	33	38	44	50	57	64	71	78	85	92
$p=$	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72
$k=$	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
$n=$	100	108	116	124	133	142	151	160	169	178	188	198	208	218	229	240	251	262
$p=$	76	80	84	88	92	96	100	104	108	112	116	120	124	128	132	136	140	144

It is clear that such values of p are optimal. As $n = 1998$ corresponds to an optimal $p = 560$, we can answer the question from which we started. For $n = 2004$ an extra cost of $4(2004-1998) = 24$ is available, but we must use edges of cost > 21. The cost of one such edge cannot be shared between all four cardinal directions. One might try to replace the (10,11) vector with (–6,17) and (16,–6), but the last is not primitive and is parallel to (8,–3) so doesn't give a new direction. But we can replace (10,11) and (1,–20) by (–1,–21), (17,–5) and (–5,17), respectively increasing the N,S,E,W components by 17–11, 21+5–20, 17–10–1, 5+1; that is, 6 in each of the four directions, and increasing the cost by $3 \cdot 22 - 2 \cdot 21 = 24$, so that $p(2004) = p(1998) - 2 + 3 = 561$.

What if n and p are large? There's a picture, Figure 5, at the end.

Theorem 330 of [2] tells us that when m is large,

$$\phi(1) + \phi(2) + \phi(3) + \cdots + \phi(m) \approx \frac{3m^2}{\pi^2}$$

so that

$$\phi(1) + 2\phi(2) + 3\phi(3) + \cdots + m\phi(m) \approx \frac{2m^3}{\pi^2}$$

and, when n is equal to the latter expression, p will be approximately 4 times the former. That is $p \approx Cn^{2/3}$ with $C = 6 \cdot 2^{\frac{1}{3}}/\pi^{\frac{2}{3}} \approx 3.524206$. If we put $n = 1998$ we get $p = 559.05987$ and $n = 2004$ gives $p = 560.17855$, both correct if we round them up. It is rare for an asymptotic expression to give such a good result! Does the error get arbitrarily large?

p or n? It may be more natural to invert the formula to

$$n \approx \frac{\pi p^{3/2}}{6\sqrt{2}}$$

and to invert the original question:

What is the smallest $n \times n$ grid that will accommodate a convex p-gon?

If p is a multiple of 4, we know the answer.

In general, the total cost of the polygon must not exceed $4(n-1)$, so write

$$4(n-1) = t + e$$

where t is the total cost of the p cheapest edges, and e is the extra expenditure that we must make. As e is non-negative, we have the following lower bounds on n for given values of p.

$p =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$t =$	0	1	2	3	4	6	8	10	12	15	18	21	24	27	30	33	36	40	44	48	52	56	60	64
$n \geq$	1	2	2	2	2	3	3	4	4	5	6	7	7	8	9	10	10	*11*	*12*	*13*	14	*15*	*16*	*17*
$p =$	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47
$t =$	68	73	78	83	88	93	98	103	108	113	118	123	128	133	138	143	148	154	160	166	172	178	184	190
$n \geq$	18	20	21	22	23	25	26	27	28	30	31	32	33	35	36	**37**	38	39	*41*	43	44	**46**	*48*	49

Figures 1 and 2 show that these minima are attained for $p \leq 16$. But, for $p = 17$, although the total cost of the p cheapest edges, $4 \cdot 1 + 4 \cdot 2 + 8 \cdot 3 + 4 = 40 = 4(11-1)$, is a multiple of 4, an 11×11 grid will not accommodate a 17-gon, because we would have to insert just one cost-4 vector into the optimal 16-gon and its cost has components 3 and 1 which can't be shared equally among the four cardinal directions. Similarly, in the cases $p = 18$, 19, 21, 22, 23, 42 and 46, the (italicized) values of n have to be increased by 1 since the components of the available edges can't be distributed equally among the four cardinal directions. In fact the 10-, 14- and 18-grids required for $p = 15$, 19 and 23 will respectively accommodate 16-, 20- and 24-gons.

Figures 3 and 4 show that the revised bounds for n can be attained for all $p \leq 48$ except for the **bold** entries under $p = 39$ and $p = 45$.

In Figure 3, $p = 13$, 14 and 15 are clear. For $p = 17$ and 18 we have omitted the 8 edges of costs 1 and 2; the labels "17=8+9" and "12=4+8" mean "8 edges omitted, 9 edges shown" and "cost 4 omitted, cost 8 shown". These polygons have been obtained by adding edges to the optimal $p = 8$, $n = 4$ solution. For $p = 21$, 22, 25 and 26, respectively enlarge $p = 16$, $n = 10$ with the pentagon, hexagon, enneagon and decagon from Figure 3.

For $p = 27$, enlarge the $p = 20$ solution which contains the edges $(\pm 1, \pm 3)$ and $(\pm 3, \mp 1)$ with the heptagon at the foot of Figure 3. and for $p = 29$, enlarge $p = 24$ with the pentagon.

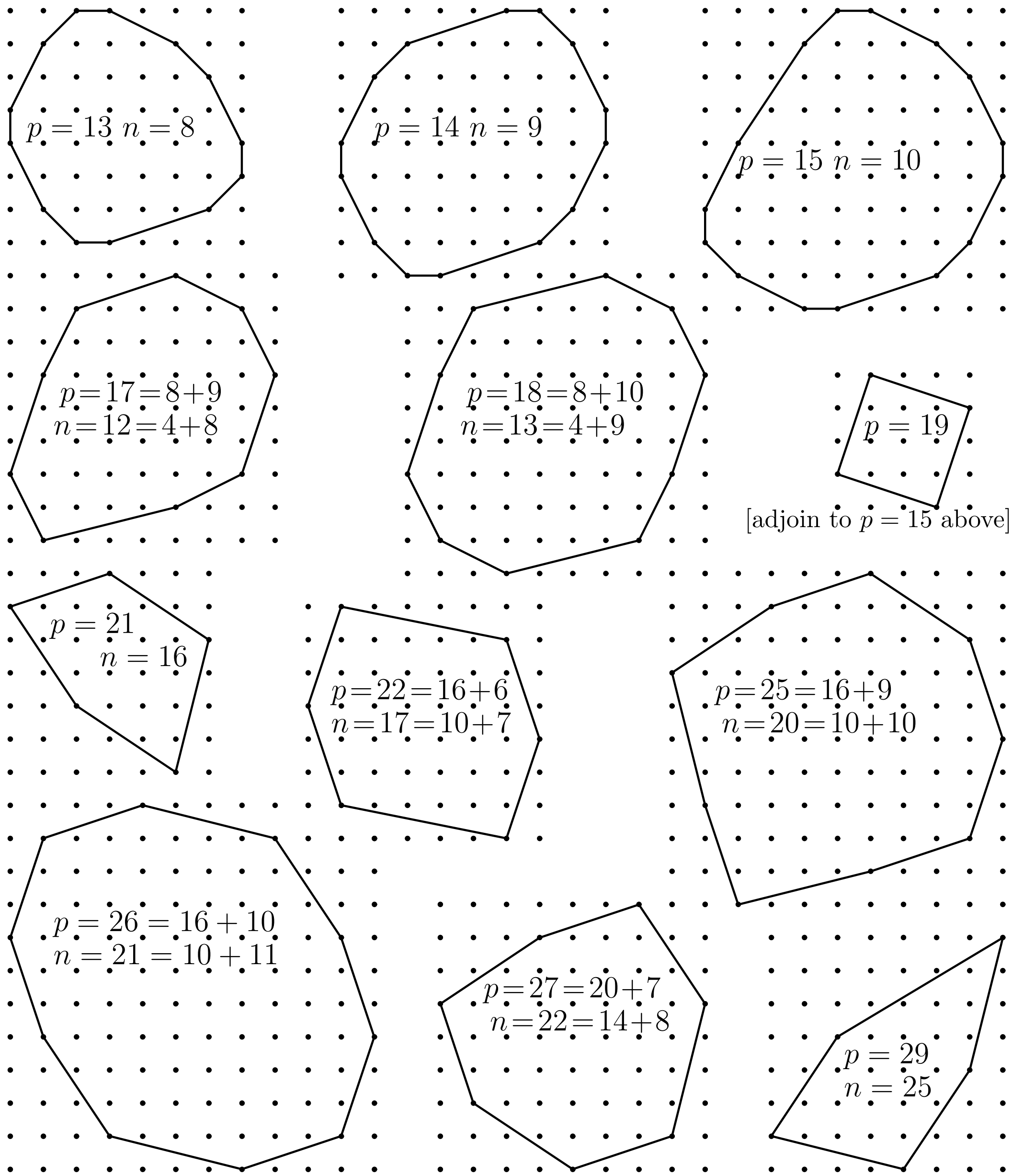

Figure 3: The smallest grids that accommodate p-gons, $13 \leq p \leq 29$.

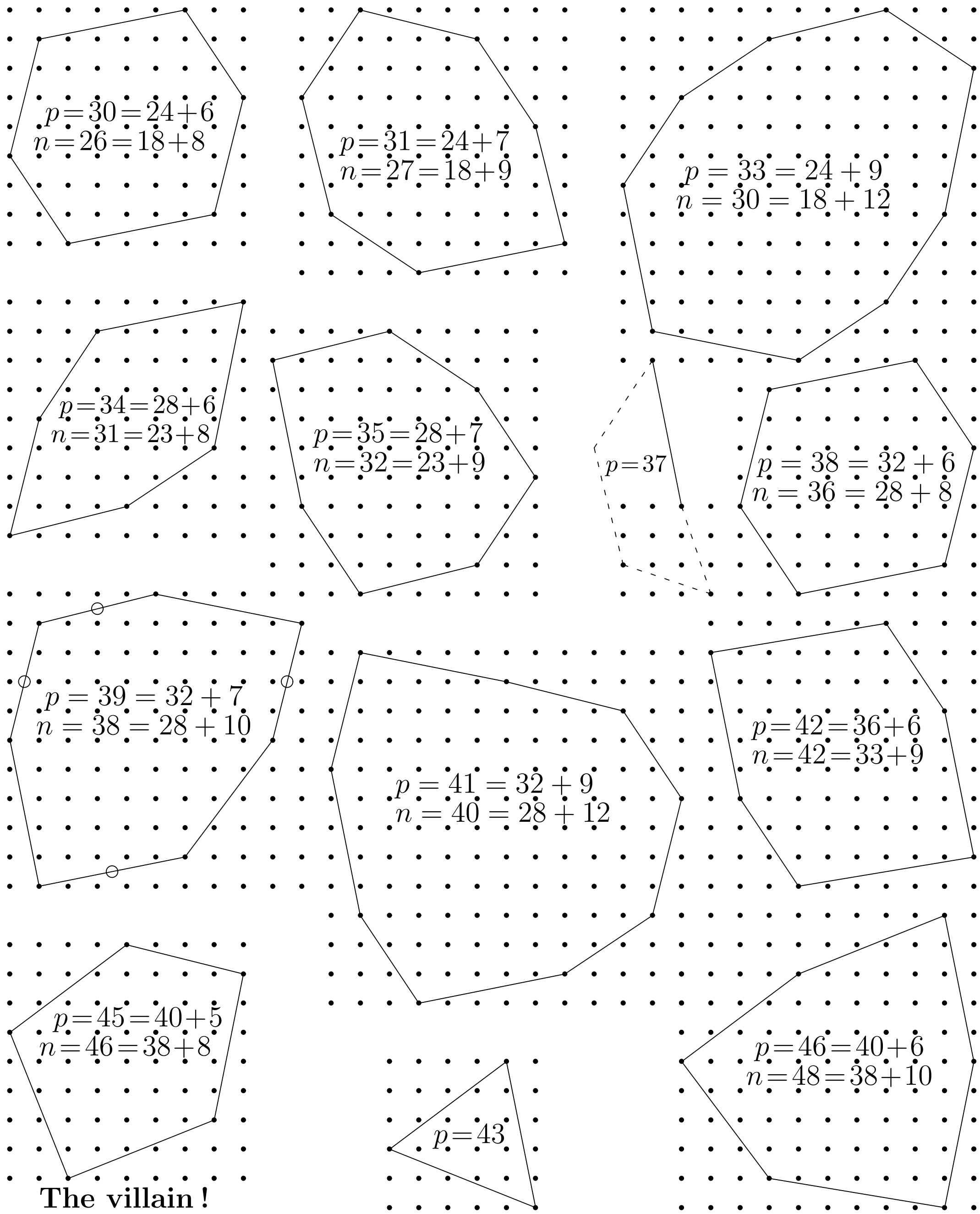

Figure 4: The smallest grids that accommodate p-gons, $30 \leq p \leq 47$, .

In Figure 4, to obtain the solutions for $p = 30$, 31 and 33, adjoin the hexagon, heptagon and enneagon to the optimal $p = 24$ solution.

Polygons $p = 34$ and $p = 35$ are got by adjoining the hexagon and heptagon to an appropriate $p = 28$ solution, where by "appropriate"we mean one which does not already contain any of the edges we propose inserting. Here, the appropriate solutions are those containing $(\pm 4, \mp 1)$ and $(\pm 1, \mp 4)$ in the first case and $(\pm 4, \mp 1)$ and $(\pm 1, \pm 4)$ in the second.

For $p = 37$, replace the 4 edges, shown dashed, in the optimal $p = 40$ solution by the edge $(-1, 5)$. For $p = 38$ adjoin the hexagon to an appropriate $p = 32$ solution, e.g., one containing $(\pm 4, \pm 1)$, $(\pm 1, \mp 4)$, $(\pm 2, \pm 3)$ and $(\pm 3, \mp 2)$.

Polygons $p = 39$ and 41 are obtained from $p = 32$; $p = 42$ from $p = 36$; and $p = 43$, 45 and 46 from $p = 40$. For 41 and 42, the appropriate $p = 32$ and $p = 36$ solutions are those containing edges $(\pm 2, \pm 3)$, $(\pm 3, \mp 2)$, $(\pm 4, \pm 1)$, $(\pm 1, \mp 4)$ in the former case and all cost-5 edges except $(\pm 3, \pm 2)$, $(\pm 2, \mp 3)$ in the latter.

Finally, $p = 47$ is obtained from $p = 40$ by using the heptagon shown for $p = 39$, but with each of the 4 edges marked with a circle stretched by cost 1.

It remains to show that for $p = 39$ and $p = 45$ we can't attain the bounds in the table. A reduced 39-gon in a 37×37 grid would have $39 = 4+4+8+8+c_5+c_6+c_7$ edges and total cost $4(37-1) = 4+8+24+32+5c_5+6c_6+7c_7$, giving $(c_5, c_6, c_7) = (14, 1, 0)$. We must replace 2 cost-5 edges from the optimal $p = 40$ solution by a cost-6 edge. But the only ways in which we can do this, e.g. replace (2,3) and (3,–2) by (5,1) yield a 39-gon in a 36×38 grid, with the correct cost, but unequally shared between E-W and N-S. In fact $p = 39$ is the first example in the pattern of places where we are just starting or just finishing using edges of cost $4k + 2$ $(k > 0)$. It's only $p = 45$ that's

The real villain. A similar calculation for $p = 45$, $n = 46$, and the requirement that $c_5 + c_7$ be even, gives the solutions $(c_5, c_6, c_7, c_8) = (14, 7, 0, 0)$, (15,5,1,0), (16,3,2,0) and (16,4,0,1), none of which can be implemented on a 46×46 grid.

Plain sailing. For $p > 48$ we shall always need to use edges of cost 7 or more, and, for $c \geq 7$, we always have $\phi(c) \geq 4$ and a good variety of slopes for our edges, and can always achieve the best bound we can hope for.

In fact we've been spooked by the Law of Small Numbers [1] and many of what have so far appeared to be exceptions are in fact part of a regular, though somewhat complicated, pattern. This is perhaps best expressed in the following summary. There's not room here for a complete proof, which tends to subdivide itself into an increasing number of cases. But the astute reader can reconstruct it from the examples that we've already seen, which are often early members of infinite sequences of polygons.

Summary. Recall that $4(n-1) = t + e$ and write $p = 4q + r$, where $r = \pm 1$ or $r = 2$, which we'll refer to as "r odd" and "r even" respectively. Write c for the cost of the most expensive of the p cheapest edges. Then

$\mathbf{e = 6}$ just if $p = 45$.

$\mathbf{e = 5}$ just if r is odd, $c = 4k + 2 + r$, **but** the cost of the $(p - \frac{1}{2}(3r-1))$-th cheapest edge is $4k+2$. Here $k > 0$. If $k = 0$, either $r = -1$, $c = 1$, $p = 3$ and the cost of the $(3 - \frac{1}{2}(-4)) = 5$-th edge $= 2$, or $r = 1$, $c = 3$, $p = 9$ and the cost of the $(9 - \frac{1}{2}(2)) = 8$-th edge $= 2$. However, $p = 3$ and $p = 9$ are the first examples of the pattern for $e = 1$ described below.

The first few examples are $p = 39$ (discussed above), $p = 49$, 111, 129, 231, 257, 383, 409, ...

$\mathbf{e = 4}$ just if $\{r$ is odd and $c = 4k\}$, or $\{r = 2$ and $c = 2k > 2\}$. ($c = 2$ is the exception $p = 6$ that we saw under $e = 0$.)

Examples that we've seen are $p = 17$, 18, 19, 21, 22, 23, 42, 46 and the next ones are $p = 73$, 74, 75, 77, 78, 79, 81, ...

$\mathbf{e = 3}$ just if r is odd and $c = 4k + r$.

Examples are $p = 1$, 11, 15, 25, 29, 33, 37 and $p = 51$, 55, 59, 63, 67, 71, 89, ...

$\mathbf{e = 2}$ just if $\{r = 2$ and c is odd$\}$ or $\{r$ is odd and $c = 4k + 2\}$, except for the villain, $p = 45$.

Examples are $p = 2$, 5, 7, 10, 14, 26, 30, 34, 38, 41, 43, 47 and $p = 50$, 54, 58, 62, 66, 70, 90, 94, 98, ...

$\mathbf{e = 1}$ just if r is odd, $c = 4k - r$, and e isn't already described above as being equal to 5.

Examples are $p = 3$, 9, 13, 27, 31, 35, but not 39, not 49, and then 53, 57. 61, 65, 69, 91, 95, 99, 103, 107, not 111, not 129, but 133, 137, ...

Reinventing the wheel. Figure 5 shows what the polygons look like when n is large. As you go from one of the 4 cardinal points to one of the intermediate ones. the radius increases in the ratio $3\sqrt{2} : 4$, or by about 6%. The curve is very close in shape to that whose equation is $x^4 + y^4 + 3(x^2 + y^2) = 4$. Its actual equation is

$$\left\{(x^2 - y^2)^2 + 6(x^2 + y^2) - 7\right\}^2 = 32(x^2 + y^2 - 1)^3$$

which is the best we can do at eliminating m from the parametric equations

$$x = \pm\frac{2m+1}{(m+1)^2} \qquad y = \pm\frac{m(m+2)}{(m+1)^2}$$

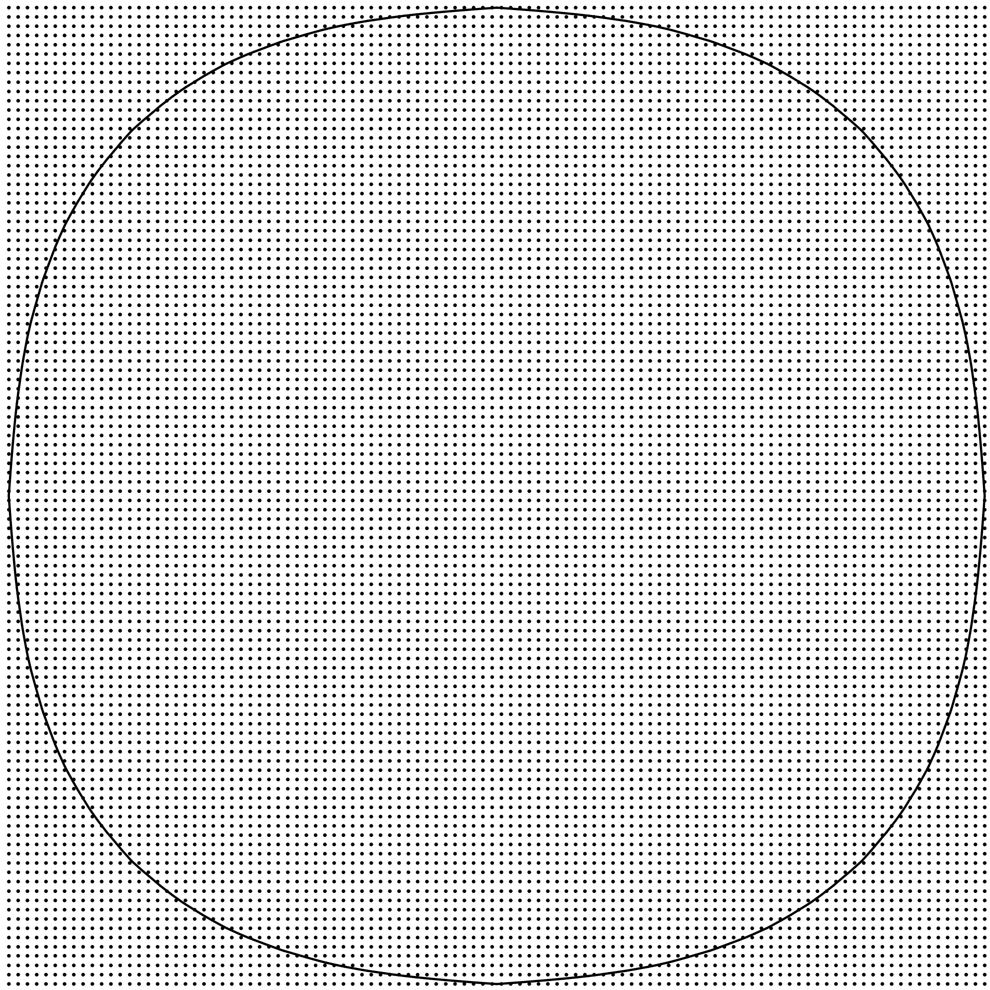

Figure 5: The 288-gon for $n = 736$ (only 106^2 lattice points shown!)

References

[1] Richard K. Guy, The strong law of small numbers, *Amer. Math. Monthly*, **95**(1988) 697–712; *MR* **90c**:11002. The second strong law of small numbers, *Math. Mag.*, **63**(1990) 3–20; *MR* **91a**:11001.

[2] G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, 4th edition, Clarendon Press, Oxford, 1960.