Valuative invariants for polymatroids

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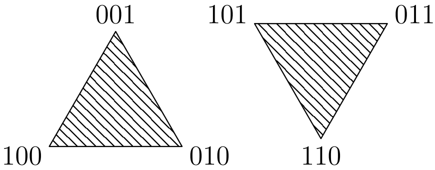
Outline

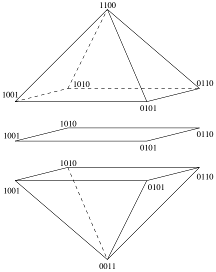
- Matroids and polymatroids
- The Tutte polynomial: a motivating example
- Valuations
- Canonical bases for (poly)matroids and valuations
- (Hopf) algebras of valuations

Definition (Edmonds; Gelfand-Goresky-MacPherson-Serganova)

A matroid M (on the ground set $[n]$) is a polytope such that

- every vertex (basis) of M lies in $\{0,1\}^n$;
- every edge of M is parallel to $e_i - e_j$ for some $i, j \in [n]$.



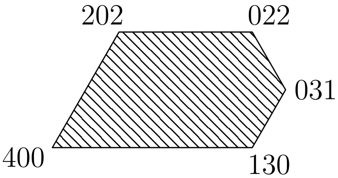


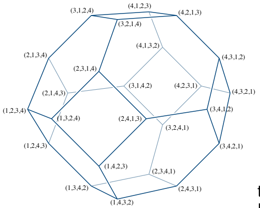
Polymatroids

Definition (Edmonds)

A polymatroid M (on $[n]$) is a polytope such that

- every vertex of M lies in $\mathbb{Z}^n_{\geq 0}$;
- every edge of M is parallel to $e_i - e_j$ for some $i, j \in [n]$.





this image
David Eppstein

Polymatroids are Postnikov's (lattice) generalised permutahedra.

Ranks

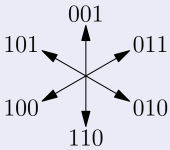
Let $e_X = \sum_{i \in X} e_i$.
The rank function of M is its support function on 0-1 vectors:

$$\mathrm{rk}_M(X) = \max_{y \in M} \langle y, e_X \rangle.$$

Fact

0-1 vectors are the only facet normals of (poly)matroids.

$$M = \{y \in \mathbb{R}^n : \langle y, e_X \rangle \leq \mathrm{rk}_M(X) \ \forall X \subseteq [n], \\ \langle y, e_{[n]} \rangle = \mathrm{rk}_M([n])\}.$$



$r := \mathrm{rk}_M([n])$ is called the rank of M.

A motivating example: the Tutte polynomial

Matroids have two operations yielding *minors*:

- deletion, $M \setminus i = \{M \cap x_i = 0\}$
- contraction, $M/i = \{M \cap x_i = 1\}$

Many invariants (e.g. # bases, independent sets, spanning sets; chromatic and flow polys of graphs; many hyperplane arr. properties; ...) can be evaluated by a deletion-contraction recurrence,

$$f(M) = f(M \setminus i) + f(M/i). \tag{1}$$

Theorem (Tutte '54, Crapo '69)

The Tutte polynomial

$$T(M; x, y) = \sum_{X \subseteq [n]} (x-1)^{r-\mathrm{rk}(X)} (y-1)^{|X|-rk(X)}$$

is universal for (1).

$$\mathbb{Z}\{\text{matroids}\}/\big(M = M \setminus i + M/i\big) = \mathbb{Z}[x, y].$$

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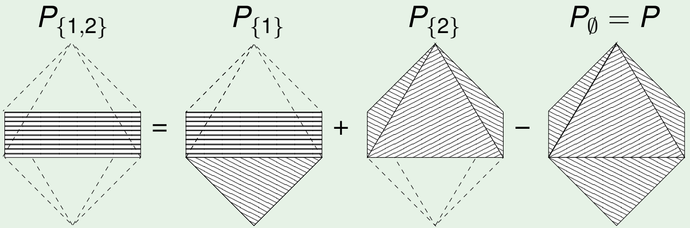
Decompositions and valuations

A decomposition $\Pi = (P; P_1, \dots, P_k)$ is a polyhedral complex. We write $P_I = \bigcap_{i \in I} P_i$.

Example



A valuation on a set $\mathcal{M}$ of polyhedra is an $f : \mathcal{M} \to G$ such that any decomposition Π with all $P_I \in \mathcal{M}$ satisfies

$$\sum_{I \subseteq [k]} (-1)^{|I|} f(P_I) = 0.$$

Examples of valuations

$$\sum_{I \subseteq [k]} (-1)^{|I|} f(P_I) = 0$$

General examples

- The map $[\cdot]$ sending P to its indicator function $[P] : \mathbb{R}^n \to \mathbb{Z}$. Many interesting evaluations, and sums and integrals of these: volume, Ehrhart polynomial, ...
- Euler characteristic χ, $\chi(P) = 1$ for $P \neq \emptyset$ if P compact.

From now on $\mathcal{M} = \{\text{matroids}\}$ or $\{\text{polymatroids}\}$.

Matroidal examples

- the Tutte polynomial T
- Speyer's invariant h, arising from K-theory of Grassmannians
- Billera-Jia-Reiner's $\mathcal{G}$, from combinatorial Hopf land

(Poly)matroid valuations

Matroid polytope decompositions come up in

- labelling fine Schubert cells in the Grassmannian (Lafforgue); connections to realisability.
- describing linear spaces via tropical geometry (Speyer, Ardila-Klivans).
- compactifying moduli of hyperplane arrangements (Hacking-Keel-Tevelev).

Problem

Describe all (poly)matroid valuations. Find a universal one.
Prove [Derksen '08]'s conjectured universal invariant $\mathcal{G}$.

Notation

Let $\mathcal{P}_\mathcal{M}$ be the $\mathbb{Z}$-module generated by indicators $[M]$ for $M \in \mathcal{M}$.
Grading: $\mathcal{P}_\mathcal{M}(r, n)$ is gen. by rank r matroids on $[n]$.

Prop'n: $\mathcal{P}_\mathcal{M}^\vee := \bigoplus \mathrm{Hom}(\mathcal{P}_\mathcal{M}(r, n), G)$ is the group of valuations.

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Bases

Define the polyhedra (full-dimensional cones)

$$R(X,\underline{r}) = \{y \in \mathbb{R}^n : \langle y, e_{X_i}\rangle \le r_i \quad \text{for } i = 1,\dots,\ell-1, \\ \langle y, e_{[n]}\rangle = r\}$$

and the (almost dual) valuations

$$s_{X,\underline{r}}(M) = \begin{cases} 1 & \text{if } \mathrm{rk}_M(X_i) = r_i \text{ for } i = 1,\dots,\ell, \\ 0 & \text{otherwise} \end{cases}$$

for $\emptyset \subsetneq X_1 \subsetneq \cdots \subsetneq X_{\ell-1} \subsetneq X_\ell = [n]$ and $\underline{r} = (r_1,\dots,r_\ell = r) \in \mathbb{Z}^\ell$.

Let $\Delta_{\mathcal{M}}(r,n)$ be the largest polyhedron in $\mathcal{M}(r,n)$.

Theorem (Derksen-F)

- *The distinct nonzero $[R(X,\underline{r}) \cap \Delta_{\mathcal{M}}(r,n)]$ form a basis for (poly)matroids mod subdivisions $\mathcal{P}_{\mathcal{M}}(r,n)$.*
- *The distinct nonzero $s_{X,\underline{r}}|_{\mathcal{M}}$ form a basis for valuations $\mathcal{P}^\vee_{\mathcal{M}}(r,n)$.*

Why these cones?

Theorem (Brianchon, Gram)

If the polyhedron P does not contain a line, then

$$[P] = \sum_F (-1)^{\dim F} [\mathrm{cone}_F(P)]$$

where F runs over all the bounded faces of P.

Proposition (Derksen-F)

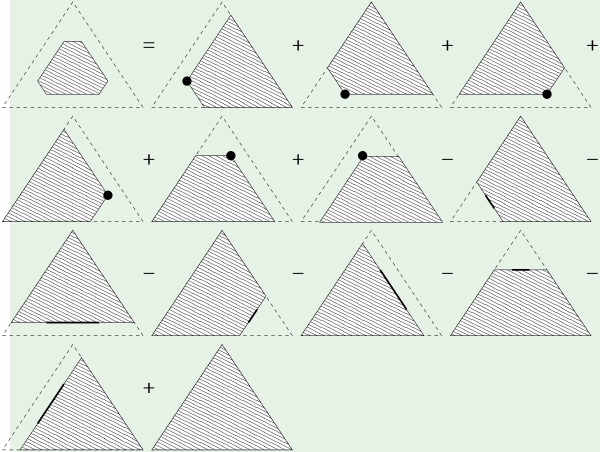
$$[M] = \sum_X (-1)^{n-\ell(X)} [R(X, \mathrm{rk}_M(X))],$$

where X ranges over all chains.

Example of the Brianchon-Gram Theorem

Example

This polytope has the combinatorial type of the permutahedron.



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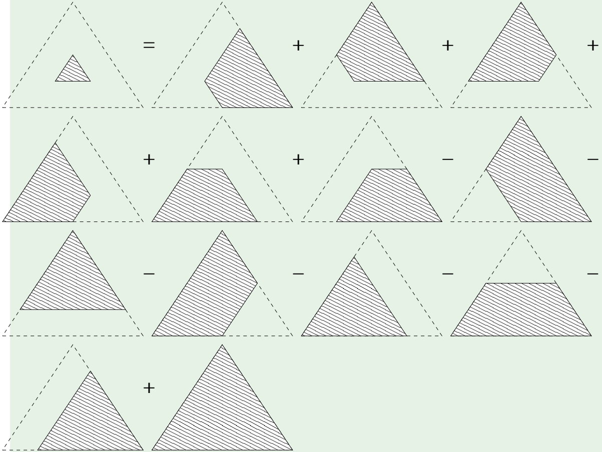
Example of the proposition

Example

We decompose this polymatroid polytope in *Rs* by inflating it to the previous one:



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- *The distinct nonzero* $s_{X,\underline{r}}|_{\mathcal{M}}$ *form a basis for valuations* $\mathcal{P}^{\vee}_{\mathcal{M}}(r,n)$.

Invariants

Our bases are unions of $\mathfrak{S}_n$-orbits.
So *unlabelled* (poly)matroids and valuative *invariants* are easy:

Theorem (Derksen-F)

- *The distinct nonzero* $[R(X,\underline{r}) \cap \Delta_{\mathcal{M}}(r,n)]$ *for a fixed maximal chain* X *form a basis for unlabelled (poly)mats mod subdivs* $\mathcal{P}_{\mathcal{M}}(r,n)/\mathfrak{S}_n$.
- *The distinct nonzero* $\sum_{X \text{ a maximal chain}} s_{X,\underline{r}}|_{\mathcal{M}}$ *form a basis for valuative invariants* $\mathcal{P}^{\vee}_{\mathcal{M}}(r,n)^{\mathfrak{S}_n}$.

The $R(X,\underline{r}) \cap \Delta_{\mathrm{Mat}}$ are exactly the polytopes of Schubert matroids.

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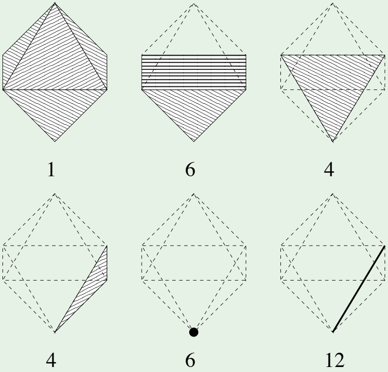
A matroid example

Example

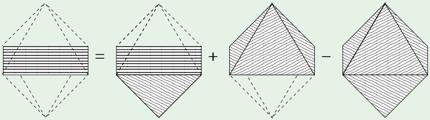
At left: one element $R(X, \underline{r}) \cap \Delta_{\mathrm{Mat}}$ of the basis of $\mathcal{P}_{\mathrm{Mat}}$ from each $\mathfrak{S}_4$-orbit, for $(n, r) = (4, 2)$.



$X = \emptyset, 1, 12, 123, 1234.$

$\underline{r} = 1222$	$\underline{r} = 1122$	$\underline{r} = 1112$
$\underline{r} = 0122$	$\underline{r} = 0012$	$\underline{r} = 0112$

Only one $\mathfrak{S}_4$-orbit of matroid polytopes isn't $\Delta_{\mathrm{Mat}} \cap$ a full-dimensional cone:

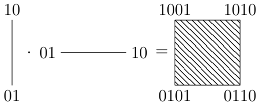


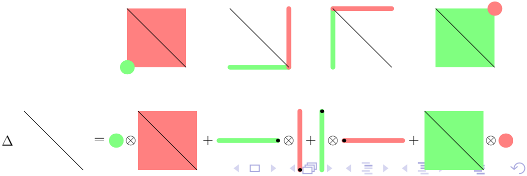
Hopf algebras of (poly)matroids

$\mathbb{Z}\mathcal{M}$, $\mathbb{Z}\mathcal{M}/\mathfrak{S}_\infty$, $\mathcal{P}_\mathcal{M}$, and $\mathcal{P}_\mathcal{M}/\mathfrak{S}_\infty$, and their duals, are Hopf algebras bigraded by (n, r).
The morphisms between them are Hopf too.

- matroids: [(Crapo-)Schmitt]
- polymatroids: [Ardila-Aguiar]

- Product is direct sum of (poly)matroids,
$$M_1 \cdot M_2 = M_1 \times M_2 = \{(m_1, m_2) : m_i \in M_i\}$$
- Coproduct is a sum over restrictions and contractions:
$$\Delta M = \sum_{X \subseteq [n]} M \setminus ([n] \setminus X) \otimes M/X$$





Hopf algebra structure of invariants

Theorem (Derksen-F)

The $\mathbb{Q}$-valued (graded) valuative invariants $(\mathcal{P}_{\mathcal{M}}^{\vee})^{\mathfrak{S}_\infty}$ form a free associative algebra:

- $\mathbb{Q}\langle u_0, u_1\rangle$ *for* $\mathcal{M} = \{$*matroids*$\}$
- $\mathbb{Q}\langle u_0, u_1, \ldots\rangle$ *for* $\mathcal{M} = \{$*polymatroids*$\}$.

We've reindexed: $u_{\underline{r}} = s_{([1],\ldots,[k]),(r_1,r_1+r_2,\ldots,r_1+\cdots+r_k)}$.

Then $u_{\underline{r}} u_{\underline{s}} = u_{\underline{rs}}$, and each u_{r_i} is primitive, $\Delta u_i = u_i \otimes 1 + 1 \otimes u_i$.

As a Hopf alg $\mathbb{Q}\langle u_0, u_1, \ldots\rangle \cong$ *NSym* is graded dual to *QSym*, the Hopf alg of quasisymmetric functions.

We get a double dual map $\mathcal{P}_{\mathcal{M}}/\mathfrak{S}_n \overset{(\sim)}{\to}$ *QSym*:

$$\mathcal{G}(M) = \sum_{\underline{r}} u_{\underline{r}}(M)\, u_{\underline{r}}^*.$$

Corollary (Derksen's conjecture)

$\mathcal{G}$ is a universal valuative invariant of (poly)matroids.

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Additive invariants

Definition

A valuation f is additive if $f(M) = 0$ whenever $\dim M < n - 1$.

So f adds on top-dimensional pieces in subdivisions.

Theorem (Derksen-F)

The additive valuative invariants form the free Lie alg $\mathbb{Q}\{u_0, u_1(, \ldots)\}$ whose universal enveloping alg is $(\mathcal{P}^{\vee}_{\mathcal{M}})^{\mathfrak{S}_n}$.

Some ingredients:
Dimension gives filtrations on our Hopf algebras.

(Poly)matroids are uniquely direct sums of connected (poly)matroids, M on $[n]$ with $\dim M = n - 1$.

$$\mathrm{gr}(\mathcal{P}_{\mathcal{M}}/\mathfrak{S}_\infty) = \mathrm{Sym}((\mathcal{P}_{\mathcal{M}}/\mathfrak{S}_\infty)_1)$$

Check one containment + enumeration.

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One intriguing future direction:

Knot diagrams can be dualised to yield graphs (i.e. matroids) with their edges (i.e. elements) two-coloured to retain crossing information.
In this setting, some known knot invariants, including the Jones polynomial, appear to become coloured matroid valuations!
Can we get new knot invariants?

Thanks for your attention!

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Thanks for your attention!

Enumeration

We get generating functions:

	$\sum \frac{\dim \mathcal{P}(r,n)}{n!} x^n y^r$	$\sum \dim \mathcal{P}(r,n)/\mathfrak{S}_n \, x^n y^r$
matroids	$\dfrac{xy - y}{xye^{-xy} - ye^{-y}}$	$\dfrac{1}{1 - xy - y}$
polymatroids	$\dfrac{e^x(1-y)}{1 - ye^x}$	$\dfrac{1-x}{1-x-y}$

In fact $\dim \mathcal{P}_{\mathrm{PMat}}(n,d)/\mathfrak{S}_n = \binom{n+d-1}{d}$ and $\dim \mathcal{P}_{\mathrm{Mat}}(n,d)/\mathfrak{S}_n = \binom{n}{d}$.

Multiplicative invariants

Definition

A function $f : \mathcal{M} \to R$ is multiplicative if $f(M_1)f(M_2) = f(M_1 \oplus M_2)$ for any (poly)matroids M_1, M_2.

Thus, f is multiplicative $\iff$ it is a group-like element of $(\mathcal{P}_{\mathcal{M}}^{\vee})^{\mathfrak{S}_\infty}$.

Example

The Tutte polynomial $T(x, y)$ is multiplicative, and

$$T = e^{(y-1)u_0+u_1} e^{u_0+(x-1)u_1}.$$