Polytopes and moduli of matroids over rings

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Perspectives in Lie Theory

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Outline

This talk is on joint work with Luca Moci, a second paper in preparation following up on `arXiv:1209.6571`.

- Matroids
- Matroids over rings, and a few applications
- The right choice of module invariants; how the axioms interrelate
- Polytopes
- The parameter space
- Coxeter generalizations?

Definition

A matroid M on the finite ground set E assigns to each subset $A \subseteq E$ a rank $\mathrm{rk}(A) \in \mathbb{Z}_{\geq 0}$, such that:

(0) $\mathrm{rk}(\emptyset) = 0$

(1) $\mathrm{rk}(A) \leq \mathrm{rk}(A \cup \{b\}) \leq \mathrm{rk}(A) + 1 \qquad \forall A \not\ni b$

(2) $\mathrm{rk}(A) + \mathrm{rk}(A \cup \{b, c\}) \leq \mathrm{rk}(A \cup \{b\}) + \mathrm{rk}(A \cup \{c\}) \qquad \forall A \not\ni b, c$

Guiding example: realizable matroids

Let $v_1, \ldots, v_n$ be vectors in a vector space V.

$$\mathrm{rk}(A) := \dim \mathrm{span}\{v_i : i \in A\}$$

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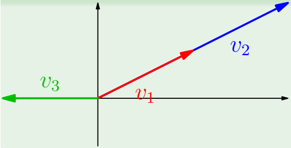
A realizable matroid in full



A	$\emptyset$	1	2	12	3	13	23	123
$\mathrm{rk}(A)$	0	1	1	1	1	2	2	2

Recast with $\mathrm{cork}(A) = r - \mathrm{rk}(A)$.

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Matroids over rings generalise matroids, as well as several variants which retain more data.

Valuated matroids come from configurations over a field with valuation, and remember valuations. [Dress-Wenzel]

Arithmetic matroids come from configurations over $\mathbb{Z}$, and remember indices of sublattices. [D'Adderio-Moci]

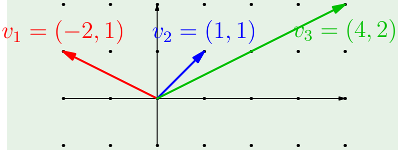
(Compare matroids with coefficients [Dress].)

Matroids over rings

Let R be a commutative ring.

Let $v_1, \ldots, v_n$ be a configuration of vectors in an R-module N.
We would like a system of axioms for the quotients $N/\langle v_i : i \in A\rangle$.

Realizable example



A	$\emptyset$	1	2	12
$M(A)$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/3$
A	3	13	23	123
$M(A)$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/2$	1

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Main definition [F-Moci]

A matroid over R on the finite ground set E assigns to each subset $A \subseteq E$ a f.g. R module $M(A)$ up to $\cong$, such that

for all $A \subseteq E$ and $b, c \notin A$, there are elements

$$x = x(b,c), \quad y = y(b,c) \in M(A)$$

with

$$M(A) = M(A), \qquad M(A \cup \{b\}) \cong M(A)/\langle x\rangle,$$
$$M(A \cup \{c\}) \cong M(A)/\langle y\rangle, \qquad M(A \cup \{b,c\}) \cong M(A)/\langle x, y\rangle.$$

Making different choices of x and y allows nonrealizability.

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Matroids are matroids over fields

Theorem 1 (F-Moci)

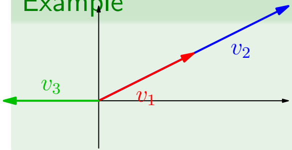
Matroids over a field $\mathbf{k}$ are equivalent to matroids*.

*if $M(E) = \emptyset$.

A f.g. $\mathbf{k}$-module is determined by its dimension $\in \mathbb{Z}$.

If $v_1, \ldots, v_n$ are vectors in $\mathbf{k}^r$,
the dimension of $\mathbf{k}^r/\langle v_i : i \in N\rangle$ is $\mathrm{cork}(A)$.

Example



A	$\emptyset$	1	2	12	3	13	23	123
$M(A)$	$\mathbb{R}^2$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	0	0	0

Note: The definition of matroids over $\mathbf{k}$ is blind to which field $\mathbf{k}$ is.
For realizability the choice matters.

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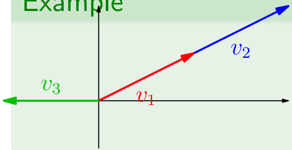

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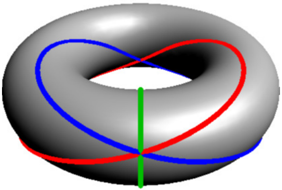
Subtorus arrangements

Let $\mathcal{H} = \{H_1, \dots, H_n\}$ be codimension one tori in an r-dim'l torus T. [De Concini-Procesi]



The subtori $H_i = \{x : u_i(x) = 1\}$ are dual to characters $u_i \in \mathrm{Char}(T) \simeq \mathbb{Z}^r$.

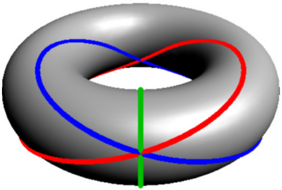
Let M be the matroid over $\mathbb{Z}$ represented by the u_i.

$$M(A) = \mathbb{Z}^k \oplus (\text{finite}) =: M(A)^{\text{free}} \oplus M(A)^{\text{torsion}}$$

Then

$$\begin{aligned} \mathrm{rk}(M(A)^{\text{free}}) &= \ \mathrm{codim} \textstyle\bigcap_{i \in A} H_i = & \dim \mathrm{span}\{u_i : i \in A\} \\ |M(A)^{\text{torsion}}| &= \ \#\ \text{components} \textstyle\bigcap_{i \in A} H_i = & [\mathbb{R}\{u_i\} \cap \mathrm{Char}(T) : \mathbb{Z}\{u_i\}] \end{aligned}$$

The arithmetic Tutte polynomial [D'Adderio-Moci] and Tutte quasipolynomial [Brändén-Moci] are invariants of M.



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Let (R, val) be a valuation ring.

Given $v_1, \ldots, v_n \in R^r$, let $p_A = \det(v_a : a \in A)$.
The ideal of relations among the p_A is generated by Plücker relations

$$p_{Abc}p_{Ade} - p_{Abd}p_{Ace} + p_{Abe}p_{Acd} = 0.$$

A valuated matroid remembers the $v_A = \mathrm{val}(p_A)$, which satisfy

$$\min\{v_{Abc} + v_{Ade}, v_{Abd} + v_{Ace}, v_{Abe} + v_{Acd}\} \text{ appears twice.}$$

The Plücker relations cut out the Grassmannian.

The valuated Plücker relations define the tropical Dressian, whose points correspond to tropical linear spaces.

A matroid over R contains the data of a tropical linear space.

But what other data is in there?

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Tropical linear spaces

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Assume (R, val) is a valuation ring.

Theorem

Any finitely presented R-module is the direct sum of copies of R and $R/\mathrm{val}^{-1}[a, \infty)$.

Let $\mathrm{length}(R) = \infty$ and $\mathrm{length}(R/\mathrm{val}^{-1}[a, \infty)\,) = a$, and extend additively.

Proposition

For a f.p. R-module N, define

$$t_i(N) := \min_{x_1,\dots,x_i \in N} \mathrm{length}(N/\langle x_1, \dots, x_i\rangle).$$

Then the series $(t_i(N))_{i \geq 0}$ is a complete isomorphism invariant.

The length axiomatization

Theorem

Let $t_i(A) \in \mathrm{val}(R) \cup \{\infty\}$ *for each* $A \subseteq E$ *and* $i \geq 0$. *There exists a matroid* M *over* R *so that* $t_i(A) = t_i(M(A)) \iff$ *for all* $A \subseteq E$ *and* $b, c \in E \setminus A$ *and* $i \geq 0$,

(Ts) *the sequence* $(t_i(A))_{i \in \mathbb{N}}$ *stabilises at zero;*

(T0) $t_i(A) - t_{i+1}(A) \geq t_{i+1}(A) - t_{i+2}(A)$;

(T1) $t_i(A) - t_{i+1}(A) \geq t_i(Ab) - t_{i+1}(Ab) \geq t_{i+1}(A) - t_{i+2}(A)$;

(T2) $t_{i+1}(A) - t_{i+1}(Ab) - t_{i+1}(Ac) + t_i(Abc) \geq \min\{t_i(Ab) - t_{i+1}(Ab), t_i(Ac) - t_{i+1}(Ac)\}$, *and equality is attained if the terms of the min differ.*

Conditions (T0–2) imply the valuated matroid axiom:

$$\min\{t_i(Abc) + t_i(Ade), t_i(Abd) + t_i(Ace), t_i(Abe) + t_i(Acd)\}$$

is attained twice. (D00)

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Let $t_i(A) \in \mathrm{val}(R) \cup \{\infty\}$ for each $A \subseteq E$ and $i \geq 0$. There exists a matroid M over R so that $t_i(A) = t_i(M(A)) \iff$ for all $A \subseteq E$ and $b, c \in E \setminus A$ and $i \geq 0$,

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Generizing and zeroizing

Let $x_1, \ldots, x_n \in N$ have matroid M over R.

If we add $x_0 = 0 \in N$ to the configuration, the new matroid M_0 has

$$M_0(A0) = M_0(A) = M(A).$$

If instead we add a suitably generic $x_* \in N$, the new matroid M_* has

$$M_*(A) = M(A), \quad t_i(M_*(A*)) = t_{i+1}(M(A)).$$

By specializing k elements to zero and ℓ to generic, $0 \leq k, \ell \leq 2$, condition (D00) becomes (D$k\ell$).

Fact

(T0) is (D22). (T1) is (D12)$\wedge$(D21). (T2) is (D11).

Almost-corollary

(D00), (D01), (D10), (Ts) are another choice of axioms.

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The (basis) polytope of a usual matroid M is

$$\mathrm{conv}\{e_A : |A| = r, \mathrm{cork}(A) = 0\}.$$

The polytope of a valuated matroid M is

$$\mathrm{conv}\{(e_A, p_A) : |A| = r\} + \mathbb{R}_{\geq 0}(\underline{0}, 1)$$

where conv discards points (v, ∞).

Theorem

P is a (valuated) matroid polytope if and only if

- *each vertex (resp. its projection to $\mathbb{R}^n$) is a 0-1 vector; and*
- *each edge (resp. its projection) is in some direction $e_i - e_j$, with $i, j \in [n]$.*

Let (R, val) be a valuation ring with $\mathrm{val}(R) \subseteq \mathbb{R}$.

For M a matroid over R, define the polytope

$$P(M) := \mathrm{conv}\{(e_A, i, t_i(M(A))\} + \mathbb{R}_{\geq 0}(\underline{0}, 0, 1).$$

Theorem

P is the polytope of a matroid over R if and only if

- *the projection of each vertex to $\mathbb{R}^n \times \mathbb{R}$ is in $\{0,1\}^n \times \mathbb{N}$;*
- *the projection of each edge is in some direction $e_i - e_j$, where $i, j \in [n] \cup \{0, n+1\}$, taking $e_0 = 0$;*
- *P contains $[0,1]^n \times [N, \infty) \times [0, \infty)$ for some N.*

If R has more primes, introduce more height coordinates.

The polytope axiomatization

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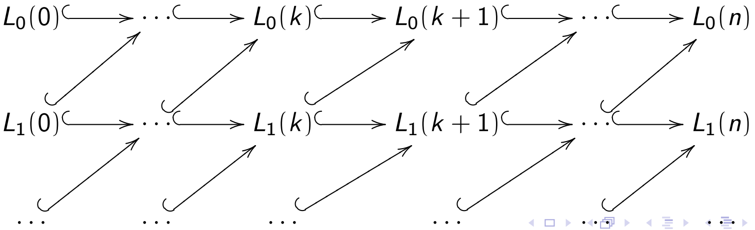
Plücker incidence relations

$$\min\{t_i(Abc) + t_i(Ad), t_i(Abd) + t_i(Ac), t_i(Acd) + t_i(Ab)\}$$
$$\text{is attained twice} \qquad \text{(D10)}$$

$$\min\{t_i(Abc) + t_{i+1}(Ad), t_i(Abd) + t_{i+1}(Ac), t_i(Acd) + t_{i+1}(Ab)\}$$
$$\text{is attained twice} \qquad \text{(D01)}$$

are the tropicalizations of incidence relations between $Gr(|A| + 1, n)$ and $Gr(|A| + 2, n)$.

If $L_i(k)$ is the tropical linear space with Plücker coordinates $t_i(M(A))$ for $A \in \binom{E}{k}$, then



(L)

The standard flag of tropical linear spaces has all Plücker coordinates zero.

Theorem

Any diagram (L) of tropical linear spaces, in which all Plücker coordinates lie in $\mathrm{val}(R)$ *and* $L_i(\cdot)$ *is the standard flag for* $i \gg 0$*, corresponds to a matroid over* R*.*

Not uniquely, since Plücker coordinates of $L_i(k)$ are nonunique.

Bott-Samelson varieties

Fix a complete flag $\mathcal{F}$ in $\mathbb{C}^n$ and let $w = s_{i_1} \ldots s_{i_\ell}$ be a word in A_{n-1}.

The Bott-Samelson variety of w is

$$Z_w = \{(\mathcal{F}_0, \ldots, \mathcal{F}_s) \in \mathcal{F}\ell_n^{s+1} : \mathcal{F}_0 = \mathcal{F}, \\ \mathcal{F}_k \text{ and } \mathcal{F}_{k+1} \text{ agree except in the } i_k\text{-dimensional space}\}.$$

Let Z_w^{trop} be its naive tropical analogue using Dressians.

Theorem

The parameter space of n-element matroids over R of global rank r is

$$\varinjlim_{k \geq 0} O_k$$

where O_k is a $r(n-r) + kn$ dim'l orthant bundle over $Z_{wc^k}^{\mathrm{trop}}$, with w a longest Grassmannian word and c a certain Coxeter element.

Tropical Schubert calculus?

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Coxeter generalizations?

Bott-Samelson varieties exist in any Weyl type:

$$Z_w = \overline{\{(x, e_{i_1}(t_1)x, \ldots, e_{i_\ell}(t_\ell)\cdots e_{i_1}(t_1)x)\}} \subseteq (G/B)^{\ell+1}$$

where the e_i are Chevalley generators.

Tropical Dressians should exist too.

Question Does this have anything to do with a valuation ring anymore?

As for $P(M)$, it has edges in directions of the A_{n+1} roots (in which an A_1 orthogonal to the A_{n-1} has a special role.)

Question $A_{n-1} : A_{n+1} :: W :$ what?

Question Are these two connected?

Thank you!

Coxeter generalizations?

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Question Are these two connected?

Thank you!

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Bott-Samelson varieties exist in any Weyl type:

$$Z_w = \overline{\{(x, e_{i_1}(t_1)x, \ldots, e_{i_\ell}(t_\ell)\cdots e_{i_1}(t_1)x)\}} \subseteq (G/B)^{\ell+1}$$

where the e_i are Chevalley generators.

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A bit more on DVRs

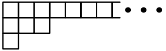
There is a bijection between

finitely generated modules over a DVR

& partitions allowing infinite parts.

Example

$N_\lambda = R \oplus R/\mathfrak{m}^3 \oplus R/\mathfrak{m}$

$\lambda =$ 

Theorem (Hall, ...)

The number of exact sequences

$$0 \to N_\lambda \to N_\nu \to N_\mu \to 0$$

up to $\cong$ of sequences is the LR coeff $c^\nu_{\lambda\mu}$ (or its infinite-rows analog).

So, quotients by one element give the Pieri rule.

Lemma, en route to Theorem 3

M is a 1-element matroid over $R \iff$
$M(\emptyset)$ has at most one box more in each column than $M(1)$.