Matroids over rings

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Outline

This talk is based on joint work with Luca Moci, `arXiv:1209.6571`.

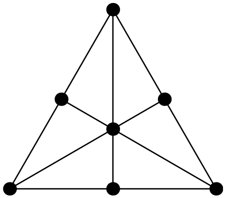
- Matroids
- Matroids over a ring
- An application of matroids, yielding matroids over $\mathbb{Z}$
- An application of matroids, yielding matroids over a DVR
- Structure, invariants

Matroids

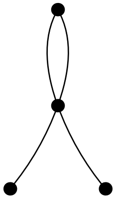
Matroids Whitney, Maclane '30s distil combinatorics from linear algebra.

An early perspective: axiomatize how (abstract) points can be contained in lines, planes, . . .

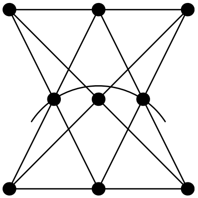
The only workable axioms are "local".



OK



Bad: $\{P, Q, R\}$ and $\{P, Q, S\}$ collinear $\Rightarrow$ all four collinear.



OK, despite Pappus! (nonrealizable)

Matroids: definition

There are lots of definitions of matroid, superficially unrelated. (Rota: "cryptomorphism".)

Definition

A matroid M on the finite ground set E assigns to each subset $A \subseteq E$ a rank $\mathrm{rk}(A) \in \mathbb{Z}_{\geq 0}$, such that: [...]

Main example: from vector configurations

Let $v_1, \ldots, v_n$ be vectors in a vector space V.

$$\mathrm{rk}(A) := \dim \mathrm{span}\{v_i : i \in A\}$$

(The v_i are our points from last slide.)

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(0) $\mathrm{rk}(\emptyset) = 0$

(1) $\mathrm{rk}(A) \leq \mathrm{rk}(A \cup \{b\}) \leq \mathrm{rk}(A) + 1 \qquad \forall A \not\ni b$

(2) $\mathrm{rk}(A) + \mathrm{rk}(A \cup \{b, c\}) \leq \mathrm{rk}(A \cup \{b\}) + \mathrm{rk}(A \cup \{c\}) \qquad \forall A \not\ni b, c$

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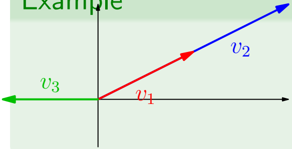
Example



A	$\emptyset$	1	2	12	3	13	23	123
$\mathrm{rk}(A)$	0	1	1	1	1	2	2	2

Now let R be a commutative ring.

Let $v_1, \ldots, v_n$ be a configuration of vectors in an R-module N.
We would like a system of axioms for the quotients $N/\langle v_i : i \in A\rangle$.

Main definition [F-Moci]

A matroid over R on the finite ground set E assigns to each subset $A \subseteq E$ a f.g. R module $M(A)$ up to $\cong$, such that

for all $A \subseteq E$ and $b, c \notin A$, there are elements $x, y \in N = M(A)$ with

$$M(A) = N, \qquad M(A \cup \{b\}) \cong N/\langle x\rangle,$$
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(1) For all $A \not\ni b$, there is a surjection $M(A) \twoheadrightarrow M(A \cup \{b\})$ with cyclic kernel.

(2) For all $A \not\ni b, c$, there are four such maps forming a pushout

$$\begin{array}{ccc} M(A) & \longrightarrow & M(A \cup \{b\}) \\ \downarrow & \lrcorner & \downarrow \\ M(A \cup \{c\}) & \longrightarrow & M(A \cup \{b, c\}) \end{array}$$

(i.e. the square commutes and $\ker \searrow = \ker \downarrow + \ker \rightarrow$)

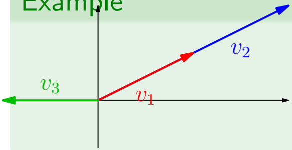
Theorem 1 (F-Moci)

Matroids over a field $\mathbf{k}$ are equivalent to matroids.

A f.g. $\mathbf{k}$-module is determined by its dimension $\in \mathbb{Z}$.

If $v_1, \ldots, v_n$ are vectors in $\mathbf{k}^r$,
the dimension of $\mathbf{k}^r / \langle v_i : i \in N \rangle$ is $r - \mathrm{rk}(A)$, the corank of A.

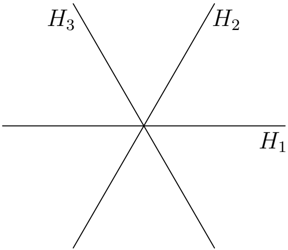
Example



A	$\emptyset$	1	2	12	3	13	23	123
$M(A)$	$\mathbb{R}^2$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	0	0	0

Application 1: hyperplane arrangement comb. & top.

Let $\mathcal{H} = \{H_1, \ldots, H_n\}$ be hyperplanes in a vector space W, $\dim W = r$.



- If W is complex, what's the cohomology $H^k(W \setminus \bigcup \mathcal{H})$?
- If W is real, how many components does $W \setminus \bigcup \mathcal{H}$ have?

$\mathcal{H}$ has a matroid: $\mathrm{rk}(A) = \mathrm{codim} \bigcap_{i \in A} H_i$.

This is also the matroid of any dual vector configuration: $(v_i \in W^\vee)$ such that

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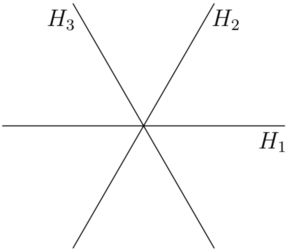

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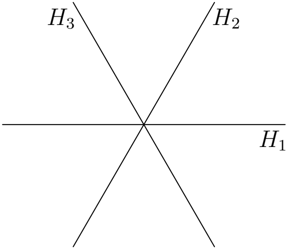
$$\chi_{\mathcal{H}}(q) = \sum_{A \subseteq E} (-1)^{|A|} q^{r - \mathrm{rk}(A)}.$$



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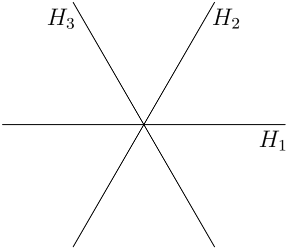
$$\sum_k \dim H^k(W \setminus \bigcup \mathcal{H}) q^k = (-q)^r \chi_{\mathcal{H}}(-1/q).$$

- $W \setminus \bigcup \mathcal{H}$ has $(-1)^r \chi_{\mathcal{H}}(-1)$ components over the reals.

The characteristic polynomial



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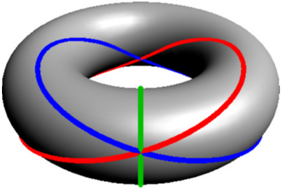
[De Concini-Procesi '10]



Subtori are dual to characters $u_i \in \mathrm{Char}(T)$:

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Here

$$\begin{aligned} \mathrm{rk}(A) &= \mathrm{codim} \textstyle\bigcap_{i \in A} H_i = && \dim \mathrm{span}\{u_i : i \in A\} \\ m(A) &= \#\text{ components } \textstyle\bigcap_{i \in A} H_i = && [\mathbb{R}\{u_i\} \cap \mathrm{Char}(T) : \mathbb{Z}\{u_i\}] \end{aligned}$$

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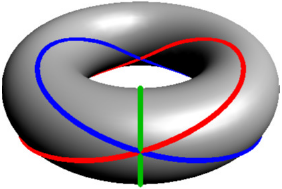

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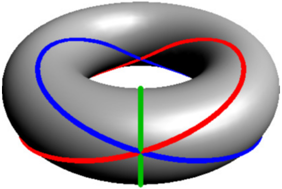
The characteristic polynomial, again

In terms of the characteristic polynomial



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Arithmetic matroids

Definition ([Moci-D'Adderio])

An arithmetic matroid is a pair (M, m), where M is a matroid and $m : 2^E \to \mathbb{Z}_{>0}$ a *multiplicity* function, such that [complicated axioms]

We have a configuration $u_i \in \mathrm{Char}(T) \cong \mathbb{Z}^r$, and:

Theorem 2 (F-Moci)

Arithmetic matroids are matroids over $\mathbb{Z}$.

Except that arithmetic matroids forget the torsion structure:

$$\mathbb{Z}^r / \langle u_A \rangle = \mathbb{Z}^{r-d} \oplus F \qquad \Longrightarrow \qquad (M(A), m(A)) = (d, |F|)$$

where F is finite.

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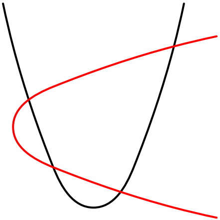
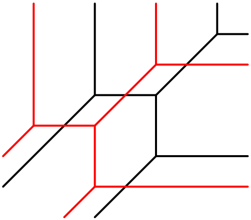
This lies among many algebro-geometric applications:

moduli of hyp arrs [Hacking-Keel-Tevelev], compactifying fine Schubert cells [Lafforgue], classes of T-orbits on Grassmannians [F-Speyer], ...

Tropical geometry studies combinatorial “shadows” of algebraic varieties.



Two conics over $\mathbb{C}$ meet in four points [Bézout]



as do two tropical conics.

Tropicalization

An algebraic variety $X \subseteq (\mathbf{k}^\times)^n$ has a tropicalization $\operatorname{Trop} X \subseteq \mathbb{R}^n$.

Suppose $(\mathbf{k}, \nu)$ has nontrivial valuation $\nu : \mathbf{k}^\times \to \mathbb{R}$, and $\mathbf{k} = \overline{\mathbf{k}}$.
Then $\operatorname{Trop} X = \overline{\nu(X)}$, coordinatewise.

Example (The line $x + y - 1 = 0$, over $\mathbb{C}[\![t^{\mathbb{Q}}]\!]$ and tropically)

If $L \subseteq \mathbf{k}^n$ is a linear space, then we tropicalize

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Tropical linear spaces

If the valuation ν is trivial, all tropicalizations are fans.

Theorem (Speyer, '04)

There is a bijection

$$\{\textit{fan tropical linear spaces}\} \longleftrightarrow \{\textit{matroids}\}$$

Definition; proposition (Dress-Wenzel '91)

A valuated matroid is a pair (M, m), where M is a matroid and $m : 2^E \to \mathbb{R}$ a *value* function, such that [axioms]. *There is a bijection*

$$\{\text{tropical linear spaces}\} \longleftrightarrow \{\text{valuated matroids}\}$$

Proposition (Speyer, '04)

$$\{\textit{tropical linear spaces}\} \longleftrightarrow \left\{\begin{array}{l}\textit{regular subdivisions}\\ \textit{of matroid polytopes}\end{array}\right\}$$

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Matroids over valuation rings

Let (R, ν) be a valuation ring.

Theorem 3 (F-Moci)

A matroid over R contains the data of a tropical linear space.

Theorem / conjecture

A matroid over R is equivalent to, for each $v \in \nu(R) \cup \{\infty\}$, a full flag of tropical linear spaces, such that [conditions].

The conjectural part is tropical moduli theory (but Haque?)
Our "full flags" tropically satisfy the Plücker relations for the full flag variety, e.g.

$$p_{A \cup b}\, p_{A \cup c, d} - p_{A \cup c}\, p_{A \cup b, d} + p_{A \cup d}\, p_{A \cup b, c} = 0$$

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The best-behaved matroids are those over Dedekind (or Prüfer) domains, i.e. rings whose localizations are (discrete) valuation rings.

Key to this is that we can tensor matroids, e.g. localize them:

$$\{\text{matroids over } R\} \xrightarrow{-\otimes_R S} \{\text{matroids over } S\}$$

A matroid M has a dual M^*. $\mathrm{rk}_M(A)$ determines $\mathrm{rk}_{M^*}(E \setminus A)$.

Example

If M comes from a vector configuration (v_i),

then M^* comes from its Gale dual: the configuration (w_i) s.t. $\{\sum_i w_{ik} v_i = 0\}$ is a basis for the linear relations among (v_i).

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If M is a matroid, $M \setminus i$ is its restriction to sets $A \not\ni i$, and
M/i is its restriction to sets $A \ni i$.

Define the Tutte-Grothendieck group to have generators $\{T_M : M \text{ a matroid}\}$ and relations

$$T_M = T_{M\setminus i} + T_{M/i}.$$

In fact it's a ring.

T_M is the Tutte polynomial of M, with many important evaluations (e.g. the characteristic polynomial).

Theorem (Crapo, Brylawski)

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Let R be a Dedekind domain.

Let $\mathbb{Z}[R\text{-Mod}]$ be the monoid ring of fin. gen. R-modules (up to $\cong$) under direct sum. $u^N u^{N'} = u^{N \oplus N'}$.

Theorem (F-Moci)

The Tutte-Grothendieck ring of matroids over R is (almost) $\mathbb{Z}[R\text{-}Mod] \otimes \mathbb{Z}[R\text{-}Mod]$, with

$$\textit{class of } M = \sum_{A \subseteq E} X^{M(A)} Y^{M^*(E \setminus A)}$$

Some specializations:

- The characteristic polynomial of a subtorus arrangement
- The Tutte quasipolynomial of [Brändén-Moci]

Future work

- Realizability?
- Other axiom systems: polytopes, circuits, . . . ?
- Connections to quotients of spheres by finite groups [Swartz]?
- . . . to flows on simplicial complexes [Chmutov et al]?
- . . . to convex hulls in Bruhat-Tits buildings [Joswig-Sturmfels-Yu]?
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