Matroids over a ring

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Outline

- Context: enriched variants of matroids
- Matroids over a ring
- Special cases
- Structure theory: local and global
- The Tutte-Grothendieck group

Enriched variants of matroids

A matroid captures the linear dependences of a vector configuration. But you might want more:

Oriented matroids come from real configurations, and remember signs (e.g. in circuits). [Bland-las Vergnas]

Complex matroids come from complex configurations, and remember phases. [Anderson-Delucchi]

Valuated matroids come from configs over a field with valuation, and remember valuations. [Dress-Wenzel]

(Quasi-)arithmetic matroids come from configurations over $\mathbb{Z}$, and remember indices of sublattices. [D'Adderio-Moci]

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Example: quasi-arithmetic matroids

“Definition”

A quasi-arithmetic matroid is a matroid with the data of an integer for each subset of the ground set, satisfying (some axioms).

Some applications: arrangements of subtori, zonotopes, box splines.

A realizable quasi-arithmetic matroid $\longleftarrow$ a vector config.
The data is the index of each sublattice in its saturation.

Matroid: uniform $U_{2,3}$

set	$\emptyset$	1	2	12
index	1	1	1	3
set	3	13	23	123
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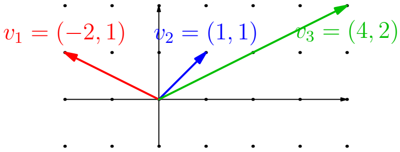

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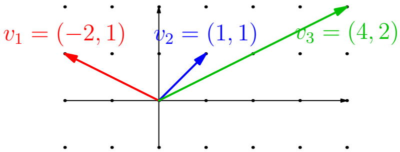
Our view of vector configurations

Why record only the *cardinality* of the torsion in the quotient?
And why separate it from the rank?

Instead, record the quotient group itself, $\mathbb{Z}^d/\langle\text{vectors}\rangle$.



set	$\emptyset$	1	2	12
group	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/3$
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group	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/2$	1

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$$V/\langle v_i : i \in A\rangle$$

is a vector space of dimension $\operatorname{corank} A$.

Fact

A corank function belongs to a matroid if every ≤ 2 element minor could come from a vector configuration.

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Matroids over rings

Let R be a commutative ring.

Definition

A matroid M over R on ground set E associates to each subset $A \subseteq E$ a finitely generated module $M(A)$, such that

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i.e. $\forall A, b, c$: there is a pushout square

$$\begin{array}{ccc} M(A) & \longrightarrow & M(A \dot\cup \{b\}) \\ \downarrow & \lrcorner & \downarrow \\ M(A \dot\cup \{c\}) & \longrightarrow & M(A \dot\cup \{b, c\}) \end{array}$$

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A trinity of special cases

Main theorem

If R is:	*then matroids over R recover:*
▶ *any field*	*(usual) matroids*
▶ *any DVR*	*valuated matroids*
▶ $\mathbb{Z}$	*quasi-arithmetic matroids*

Dedekind domains

From now on R is a Dedekind domain, i.e. a regular one-dim'l ring.

Review: structure theory of R-modules

Every R-module uniquely has the form

$$\underbrace{\left.\begin{array}{rc} & R^{c-1} \oplus P \\ \text{or} & 0 \end{array}\right\}}_{\text{projective}} \oplus \underbrace{\bigoplus R/\mathfrak{m}_i^{a_i}}_{\text{torsion}}$$

for P a rank 1 projective module, $\mathfrak{m}_i$ maximal ideals, $a_i > 0$ integers.

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Structure of matroids over a Dedekind domain

You can do base changes (e.g. localization) on matroids over rings:

$$\{\text{matroids over } R\} \xrightarrow{-\otimes_R S} \{\text{matroids over } S\}$$

Strategy

To understand matroids over a Dedekind domain R:

1. What can their localizations be like?
2. When does a family of localizations come from a global matroid?

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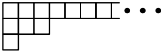
Local structure: modules over a DVR

TFAIB:	Example
finitely generated modules over a DVR	$N_\lambda = R \oplus R/\mathfrak{m}^3 \oplus R/\mathfrak{m}$
partitions allowing infinite parts	$\lambda =$ 
nonincreasing sequences in $\mathbb{N}$	$d(N_\lambda) = 3, 2, 2, 1, 1, 1, 1, \ldots$

Theorem (Hall, . . .)

If λ, μ, ν have finite parts, the number of exact sequences

$$0 \to N_\lambda \to N_\nu \to N_\mu \to 0$$

up to isomorphism is the Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$.

Cyclic kernel $\Longrightarrow$ Pieri rule.

Local structure: matroids over a DVR

If N is an R-module, let $d_n(N) = \#$ boxes in column n,
(n may be ∞) $d_{\leq n}(N) = \#$ boxes in or left of column n.

Theorem

M is a 1-element matroid over $R \iff d_n(M(1)) - d_n(M(\emptyset)) \in \{0,1\}$.

Theorem

M is a 2-element matroid over $R \iff$ further, $d_{\leq n}$ is supermodular:

$$d_{\leq n}(M(\emptyset)) + d_{\leq n}(M(12)) \geq d_{\leq n}(M(1)) + d_{\leq n}(M(2)),$$

and equality is attained if $d_n(M(1)) \neq d_n(M(2))$.

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M is a 3-element matroid over $R \implies$ the minimum among

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The last theorem says that, if $p_A := d_{\leq n}(M(A))$, the Plücker relation

$$p_{Ab}p_{Acd} - p_{Ac}p_{Abd} + p_{Ad}p_{Abc} = 0$$

for the full flag variety vanishes tropically.

Conjecture

The vector of $d_{\leq n}(M(A))$ for all A defines a point on the tropical full flag variety*.

Theorem

The vector of $d_{\leq n}(M(A))$ defines a point on each tropical Grassmannian.*

Equivalently, $d_{\leq n}(M(A))$ is a valuated matroid.

* Tropical experts: I really mean Dressians.

The Tutte-Grothendieck group has generators $\{T_M : M \text{ a matroid}\}$ and relations

$$T_M = T_{M\backslash i} + T_{M/i}.$$

In fact it's a ring, with $T_M T_{M'} = T_{M\oplus M'}$.

T_M is the Tutte polynomial of M.

Theorem (Crapo, Brylawski)

The Tutte-Grothendieck ring is $\mathbb{Z}[x-1, y-1]$, with

$$T_M = \sum_{A\subseteq E} (x-1)^{\operatorname{corank}_M(A)} (y-1)^{\operatorname{corank}_{M^*}(E\backslash A)}$$

Let S be the monoid ring of fin. gen. R-modules (up to $\cong$) under direct sum.

Theorem

The Tutte-Grothendieck ring of matroids over R is essentially $S \otimes S$, with

$$\text{class of } M = \sum_{A \subseteq E} X^{M(A)} Y^{M^*(E \setminus A)}$$

Why "essentially"? $M(A)$ and $M^*(E \setminus A)$ must have the same torsion part.

One specialization: Brändén-Moci's Tutte quasipolynomial.

Looking ahead

Are matroids over rings relevant to

- Chmutov's "arithmetic flow quasipolynomial" of simplicial complexes (over $\mathbb{Z} \oplus \mathbb{Z}$)?
- point configurations in type A Bruhat-Tits buildings (over a DVR)?

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Thank you!

Why Dedekind domains?

Regularity makes the projective modules and K-theory well-behaved.

One-dimensionality makes our maps essentially unique:

Fact

If R is a Dedekind domain, then given two R-modules M, N, all cyclic kernels of surjections $M \twoheadrightarrow N$ are isomorphic.

That is, the Pieri rule is coefficient-free.

Counterexample in dimension 2

Two surjections between two $\mathbf{k}[x, y]$-modules with nonisomorphic cyclic kernels: