Integral representation and critical L-values for the standard L-function of a Siegel modular form

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Special value results

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In other words, the special values of the Riemann zeta function at positive even integers are rational numbers up to a power of π.

$1 - \frac{1}{3} + \frac{1}{5} - \ldots = \frac{\pi}{4}$, and more generally

$$\frac{1 - \frac{1}{3^{2m-1}} + \frac{1}{5^{2m-1}} - \ldots}{\pi^{2m-1}} = \frac{L(2m-1, \chi_4)}{\pi^{2m-1}} \in \mathbb{Q},$$

where χ_4 is the unique Dirichlet character of conductor 4.

An example with modular forms

The Ramanujan Δ-function is defined as follows:

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty}(1 - e^{2\pi inz})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz}.$$

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Theorem (Manin, Shimura)

There exist real numbers r^+, r^- such that

- $$\frac{L(m, \Delta)}{r^- \cdot \pi^m} \in \mathbb{Q} \quad \text{for } m = 1, 3, 5, 7, 9, 11$$
- $$\frac{L(m, \Delta)}{r^+ \cdot \pi^m} \in \mathbb{Q} \quad \text{for } m = 2, 4, 6, 8, 10$$

Shimura's result on the Rankin–Selberg L-function

Let $k > l$. Let $f \in S_k(M)$, $g \in S_l(M)$

$$f(z) = \sum_{n>0} a_n e^{2\pi i n z}, \quad g(z) = \sum_{n>0} b_n e^{2\pi i n z}$$

$$L(s, f \otimes g) = \zeta(2s) \sum_{n=1}^{\infty} \frac{a_n b_n}{n^{\frac{k+l-2}{2}+s}}.$$

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Shimura, 1976

For $m \in \frac{1}{2}\mathbb{Z}$, define

$$C(m, f, g) = \frac{L(m, f \otimes g)}{\pi^{2m+k}\langle f, f\rangle}.$$

Then, for $m \in \left[\frac{1}{2}, \frac{k-l}{2}\right] \cap \left(\mathbb{Z} + \frac{k+l}{2}\right)$, we have $C(m, f, g) \in \overline{\mathbb{Q}}$

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Similar results can be proved for convolutions of three cusp forms (Sturm, Garrett-Harris,..).

All these special value results fall under a broad conjecture due to Deligne :

Deligne's conjecture

Let $L(s, M)$ be the L-function associated to a "motive". Then there exists a set S of "critical points" such that for all $m \in S$,

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Results in the spirit of Deligne are were studied by Shimura in the 1970s for classical holomorphic modular forms of arbitrary weight and character, and their Rankin-Selberg convolutions.

What about more general L-functions

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2. Classical modular forms correspond to automorphic representations of GL_2. Natural next case: Siegel modular forms.
3. Siegel modular forms of degree n correspond to automorphic representations of GSp_{2n}. Of great interest for many reasons, e.g., paramodular conjecture: L-functions of rational abelian surface are L-functions of certain Siegel cusp forms of degree 2.

Siegel modular forms of degree n

Definition of Sp_{2n}

For a commutative ring R, we denote by $\mathrm{Sp}_{2n}(R)$ the set of $2n \times 2n$ matrices $A \in \mathrm{GL}_{2n}(R)$ satisfying the equation $A^t J A = J$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

Definition of $\mathbb{H}_n$

Let $\mathbb{H}_n$ denote the set of complex $n \times n$ matrices Z such that $Z = Z^t$ and $\mathrm{Im}(Z)$ is positive definite.

$\mathbb{H}_n$ is a homogeneous space for $\mathrm{Sp}_{2n}(\mathbb{R})$ under the action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto (AZ+B)(CZ+D)^{-1}$$

The space $S_\rho(\Gamma)$

Let Γ be a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$ and (ρ, V) a representation of $\mathrm{GL}_n(\mathbb{C})$.

Siegel modular forms

A holomorphic vector valued Siegel modular form of degree n, level Γ and weight ρ is a holomorphic V-valued function F on $\mathbb{H}_n$ satisfying

$$F(\gamma Z) = \rho(CZ + D)F(Z),$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$,
If in addition, F vanishes at the cusps, then F is called a cusp form.

We define $S_\rho(\Gamma)$ to be the space of cusp forms as above.

Let (ρ, V) be a representation of $\mathrm{GL}_n(\mathbb{C})$.

- Let $F \in S_\rho(\Gamma)$ be a Hecke eigenform. $F \leftrightarrow \pi$, an automorphic representation of $\mathrm{GSp}_{2n}(\mathbb{A})$.
- Note: $\pi_\infty \Leftrightarrow (k_1, k_2, \ldots, k_n)$ of non-increasing integers. $\dim(V) = 1 \Leftrightarrow k_1 = k_2 = \ldots = k_n$. This is the scalar valued case.

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  2. For $\dim(V) > 1$, only very special cases have been done, namely $V = \det^\ell \mathrm{sym}^m$ AND $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$.
  3. Key stumbling block: Sufficiently nice integral representation (without any restriction on level Γ or ρ)

Main results (Pitale - S -Schmidt)

1. An explicit integral representation for $L(s, \pi \boxplus \chi)$ for all representations π of $\mathrm{GSp}_{2n}(\mathbb{A})$, wth $\pi_\infty = (k_1, k_2, \ldots, k_n)$, all k_i same parity, $\chi(-1) = (-1)^{k_1}$.
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   - Key new idea: An explicit choice of vectors in the pullback formula/doubling method which involves a nearly holomorphic modular form, and allows us to exactly compute all local integrals.
2. By specializing to case $n = 2$, get algebraicity of critical values for $L(s, \pi \boxplus \chi)$.
   - Key ingredient: A structure theorem for the space of nearly holomorphic modular forms of degree 2.
   - By specializing further, get an application to critical values for the symmetric fourth of a modular form!

Our result on special values in classical language

Theorem 1 (Pitale-S-Schmidt)

Let $\rho = \det^l \mathit{sym}^m$*, m even.* $F \in S_\rho(\Gamma)$ *be an eigenfunction of Hecke operators, with algebraic Fourier coefficients. Then for all even integers r,* $1 \leq r \leq l-2$, $r \equiv l \mod 2$, $\chi(-1) = (-1)^l$,

$$\frac{L_{st}(r, F, \chi)}{(2\pi i)^{2k+3r} G(\chi)^3 \langle F_0, F_0 \rangle} \in \overline{\mathbb{Q}}$$

and moreover this ratio behaves nicely under actions of $Aut(\mathbb{C})$.

- F_0 is the nearly holomorphic form associated to F
- Generalizes previously known special value results for this L-function.

An application to sym^4

Let $f \in S_k(N)$ be a classical newform. $\chi(-1) = (-1)$.

Theorem 2 (Pitale-S-Schmidt)

Let $1 \le r \le k-1$ be odd. Then

$$\frac{L(r, \chi \otimes \mathrm{sym}^4 f)}{(2\pi i)^{4k-2+3r} G(\chi)^3 \langle F_0, F_0 \rangle} \in \overline{\mathbb{Q}}$$

and moreover this ratio behaves nicely under actions of $Aut(\mathbb{C})$.

- Uses the sym^3 lift of Ramakrishnan-Shahidi.
- Previously a result on sym^4 L-function was known for f of full level (Ibukiyama-Katsurada).
- Result on sym^n, n odd, known assuming functoriality (Raghuram).

- "Lowest weight modules of $\mathrm{Sp}_4(\mathbb{R})$ and nearly holomorphic Siegel modular forms", Pitale-Saha-Schmidt, **arXiv:1501.00524**, 2015.
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- Some possible future directions: general degree n, congruences, etc.

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Thank you for listening!