[*This document is the more detailed version of my paper that was published in* International Math. Research Notices IMRN (2009), Number 10, Pages 1773–1837.]

L-FUNCTIONS FOR HOLOMORPHIC FORMS ON $GSp(4) \times GL(2)$ AND THEIR SPECIAL VALUES

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Abstract. We provide an explicit integral representation for L-functions of pairs (F, g) where F is a holomorphic genus 2 Siegel newform and g a holomorphic elliptic newform, both of squarefree levels and of equal weights. When F, g have level one, this was earlier known by the work of Furusawa. The extension is not straightforward. Our methods involve precise double-coset and volume computations as well as an explicit formula for the Bessel model for $GSp(4)$ in the Steinberg case; the latter is possibly of independent interest. We apply our integral representation to prove an algebraicity result for a critical special value of $L(s, F \times g)$.

Introduction

L-functions for automorphic forms on reductive groups are objects of considerable number theoretic interest. They codify the relationship between arithmetic and analytic objects and enable us to investigate properties that are otherwise not easily accessible. One of the tools that has been successfully used to study L-functions and their special values is the method of integral representations; this is sometimes called the Rankin-Selberg method after Rankin and Selberg's fundamental work in this direction. Often, sharper and more explicit results are obtained when one restricts attention to holomorphic forms. The papers [4], [5], [9] treating the triple-product L-function, are good examples, and in fact, provided an inspiration for this article.

Let $\pi = \otimes \pi_v$, $\sigma = \otimes \sigma_v$ be irreducible, cuspidal automorphic representations of $GSp_4(\mathbb{A})$, $GL_2(\mathbb{A})$ respectively, where $\mathbb{A}$ denotes the ring of adeles over $\mathbb{Q}$. In this paper we are interested in the degree eight L-function $L(s, \pi \times \sigma)$. Furusawa [2] discovered an integral representation for this L-function in the case when π and σ are both unramified at all finite places. However, for several applications, this case is not enough. To give an example, suppose $F = \mathrm{Sym}^3(E_1)$ is a holomorphic Siegel cusp form that arises as the symmetric cube of an elliptic curve E_1 over $\mathbb{Q}$ (as worked out by Ramakrishnan-Shahidi in [19]) and g is an holomorphic elliptic cusp form associated to another elliptic curve E_2 over $\mathbb{Q}$. Then neither F, nor g, can be of full level (since there exists no elliptic curve over $\mathbb{Q}$ that is unramified everywhere). Furthermore, the local components of the representations associated to F and g at all ramified places are Steinberg. So, if we wish to study $L(s, F \times g)$ in this case, we would need to evaluate the local zeta integral when one or both of the local representations is Steinberg.

In order to state the results of this paper, we first recall the integral representation of [2] in detail.

Let L be a quadratic extension of $\mathbb{Q}$ and consider the unitary group $GU(2,2) = GU(2,2;L)$. Let P be the maximal parabolic of $GU(2,2)$ with a non-abelian unipotent radical. Note that GL_2 embeds naturally inside a Levi component of P. So, given an automorphic representation σ of $GL_2(\mathbb{A})$ and a Hecke character Λ of L we can form an automorphic representation Π on $P(\mathbb{A})$ and thus an induced representation $I(\Pi, s) = \mathrm{Ind}_{P(\mathbb{A})}^{GU(2,2)(\mathbb{A})}(\Pi \times \delta_P^s)$. In the usual manner we then define an Eisenstein series $E(g, s; f)$ on $GU(2,2)(\mathbb{A})$ for an analytic section $f \in I(\Pi, s)$.

For an vector Φ in the space of π and an analytic section $f \in I(\Pi, s)$ consider the global integral

$$Z(s) = \int_{Z(\mathbb{A})GSp_4(\mathbb{Q})\backslash GSp_4(\mathbb{A})} E(g, s; f)\overline{\Phi(g)}dg. \tag{0.0.1}$$

In [2], Furusawa proves the following results:

(a) For suitable choices of L, Λ and f, $Z(s)$ is Eulerian, that is

$$Z(s) = \prod_v Z_v(s)$$

where for each place v of $\mathbb{Q}$, $Z_v(s)$ is an explicit local zeta integral.

(b) Let p be a finite prime where both π_p and σ_p are unramified. Then

$$Z_p(s) = C(s) \times L(3s + \frac{1}{2}, \pi_p \times \sigma_p),$$

where $C(s)$ is an explicit normalizing factor.

We now state the main local result of this paper. For the more precise version, see the Theorems 5.3.1, 6.3.1, 7.3.1.

Theorem A. *Let p be a finite prime which is inert in L.*

(a) *Suppose that π_p is unramified and σ_p is an unramified quadratic twist of the Steinberg representation. Also suppose that Λ_p is unramified. Then we have*

$$Z_p(s) = \frac{1 - p^{-6s-3}}{p^2 + 1} \times L(3s + \frac{1}{2}, \pi_p \times \sigma_p).$$

(b) *Suppose that π_p is an unramified quadratic twist of the Steinberg representation and σ_p is unramified. Also suppose that Λ_p has conductor p. Then we have*

$$Z_p(s) = \frac{1}{(p+1)(p^2+1)} \times L(3s + \frac{1}{2}, \pi_p \times \sigma_p).$$

(c) *Suppose that π_p , σ_p are both unramified quadratic twists of the Steinberg representations. Also suppose that Λ_p has conductor p. Then we have*

$$Z_p(s) = \frac{p^{-6s-3}}{p(p^2+1)(1 - a_p w_p p^{-3s-\frac{3}{2}})} \times L(3s + \frac{1}{2}, \pi_p \times \sigma_p),$$

where a_p is the eigenvalue of the local operator T_p for σ_p and w_p is the eigenvalue of the local Atkin-Lehmer operator for π_p.

As already noted, the simplest case where both local representations are unramified was proved in [2]. However the methods employed for that case are not sufficient to deal with the above three cases. The explicit evaluation of the local zeta integral involves several steps. First of all, we need to perform certain technical volume and double-coset computations. These computations — easy in the unramified case — are tedious and challenging for the remaining cases and are carried out in Section 3. Secondly, it is necessary to suitably choose the sections of the Eisenstein series at the bad places to insure that the local zeta integrals do not vanish. Thirdly, and perhaps most crucially, the local computations require an explicit knowledge of the local Whittaker and Bessel functions. The formulae for the Whittaker model are well known in all cases; the same, however, is not true for the Bessel model. In fact, the only case where the local Bessel model for a finite place was computed before this work was when π_p is unramified [23]. However, that does not suffice for the two cases when we have π_p Steinberg. As a preparation for the calculations in these cases, we find, in Section 4, *an explicit formula for the Bessel function for π_p when it is Steinberg.* This is perhaps of independent interest.

Putting together our local computations we get an integral representation for the global case as described next.

For a square-free integer M let $B(M)$ denote the minimal (Iwahori) congruence subgroup of level M in $Sp(4,\mathbb{Z})$ consisting of matrices of the form

$$Sp(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & N\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

We say that a holomorphic Siegel cusp form of genus 2 is a *newform of level M* if:

(a) It lies in the orthogonal complement of the space of oldforms for $B(M)$ as defined by Schmidt [21].
(b) It is an eigenform for the Hecke algebra at all primes not dividing M.
(c) It is an eigenform for the Atkin-Lehner operator at all primes dividing M.

For a square-free integer N, we call a holomorphic elliptic cusp form is a newform of level N if it is a newform with respect to the group $\Gamma_0(N)$ in the usual sense.

Now, fix odd, square-free positive integers M, N and let F be a genus 2 Siegel newform of level M and g an elliptic newform of level N. We assume that F and g have the same even integral weight l and have trivial central characters. Furthermore we make the following (mild) assumption about F:

Suppose

$$F(Z) = \sum_{S>0} a(S)e(\mathrm{tr}(SZ))$$

is the Fourier expansion; then we assume that

$$a(T) \neq 0 \text{ for some } T = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \tag{0.0.2}$$

such that $-d = b^2 - 4ac$ is the discriminant of the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$, and all primes dividing MN are inert in $\mathbb{Q}(\sqrt{-d})$.

Let Φ denote the adelization of F. The representation of $GSp(4)(\mathbb{A})$ generated by Φ may not be irreducible, but we know [21] that all its irreducible components are isomorphic. Let us denote any of these components by π. Also, we know that g generates an irreducible representation σ of $GL_2(\mathbb{A})$. We prove (see Theorem 8.5.1 for the full statement) the following result:

Theorem B. *Let F, g, π, σ be as defined above. Then, for a suitable choice of Λ, f, the global integral defined in* (0.0.1) *satisfies*

$$Z(s) = C(s) \times L(3s + \frac{1}{2}, \pi \times \sigma),$$

where $C(s)$ *is an explicit normalizing factor.*

Using the above integral representation, one can prove a certain special value result. Before stating that, we make some general remarks. If $L(s)$ is an arithmetically defined (or motivic) L-series, it is interesting to study it's value at certain critical points $s = m$. For these critical points, the standard conjectures predict that $L(m)$ is the product of a suitable transcendental number Ω and an algebraic number $A(m)$. Moreover, it is expected that the same Ω works for $L_\chi(m)$ where χ is a Dirichlet character of appropriate parity.

As a consequence of Theorem B, we get, using a theorem of Garrett [3], the following special value result. This fits into the framework of the conjectures mentioned above.

Theorem C. *Suppose F, g are as defined above and moreover have totally real algebraic Fourier coefficients. Then, assuming $l > 6$, we have*

$$\frac{L(\frac{l}{2} - 1, F \times g)}{\pi^{5l-8}\langle F, F\rangle\langle g, g\rangle} \in \overline{\mathbb{Q}} \tag{0.0.3}$$

where $\langle\ \rangle$ denotes the Petersson inner product.

We should note that Theorem C has been previously proved in the basic case $M = 1, N = 1$ by Furusawa [2] and (independently) by Bernard Heim [10], who used a different integral representation. After this paper was essentially complete, it was brought to the attention of the author that Pitale and Schmidt [18] have independently, and around the same time, evaluated the local Furusawa integral above in the case when π_p is unramified but σ_p is Steinberg. This allows them to prove analogues of Theorem B and Theorem C in the case $M = 1, N \geq 1$ square-free.

This paper, to our best knowledge, is the first that gives an integral representation or proves any special value result for $L(s, F \times g)$ when $M > 1$.

It is of interest to find, in addition, a reciprocity law relating to the above special value, that is, the equivariance of the action of $\mathrm{Aut}(\mathbb{C})$ on the quantity defined in (0.0.3). Unfortunately, not enough is known about the corresponding action on the Fourier coefficients of our Eisenstein series to resolve this question here. In a sequel to this paper [20], we will use a certain pullback formula to get another integral representation for our L-function that involves a well-understood Siegel Eisenstein series on $GU(3, 3)$. That will enable us to answer the $\mathrm{Aut}(\mathbb{C})$ equivariance and related questions. We say a little more about these techniques in the final section of this paper.

Acknowledgements. The ideas and methods employed in this paper draw on various sources and it would be impossible to mention them all. However, as already mentioned, the author was influenced by the work of Gross and Kudla [5] on the

triple product L-function. This paper also owes an obvious debt to the fundamental work of Furusawa [2].

The author also thanks M. Harris for some helpful suggestions (whose importance will be more apparent in the sequel to this paper), D. Lanphier and A. Pitale for useful discussions and T. Tsankov for proof-reading a part of this paper.

This work was done while the author was a graduate student at Caltech and represents part of his PhD. dissertation. The author thanks his advisor Dinakar Ramakrishnan for guidance, support and many helpful discussions.

Notation.

- The symbols $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ have the usual meanings. $\mathbb{A}$ denotes the ring of adeles of $\mathbb{Q}$. For a complex number z, $e(z)$ denotes $e^{2\pi iz}$.

- For any commutative ring R and positive integer n, $M_n(R)$ denotes the ring of n by n matrices with entries in R and $GL_n(R)$ denotes the group of invertible matrices in $M_n(R)$. If $A \in M_n(R)$, we let A^T denote its transpose. We use $R^\times$ to denote $GL_1(R)$.

- Denote by J_n the $2n$ by $2n$ matrix given by
$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$
We use J to denote J_2.

- For a positive integer n define the group $GSp(2n)$ by
$$GSp(2n, R) = \{g \in GL_{2n}(R) | g^T J_n g = \mu_n(g) J_n, \mu_n(g) \in R^\times\}$$
for any commutative ring R.
Define $Sp(2n)$ to be the subgroup of $GSp(2n)$ consisting of elements $g_1 \in GSp(2n)$ with $\mu_n(g_1) = 1$.
The letter G will always stand for the group $GSp(4)$ and G_1 for the group $Sp(4)$.

- For a commutative ring R we denote by $I(2n, R)$ the Borel subgroup of $GSp(2n, R)$ consisting of the set of matrices that look like $\begin{pmatrix} A & B \\ 0 & \lambda(A^T)^{-1} \end{pmatrix}$. where A is lower-triangular and $\lambda \in R^\times$. Denote by B the Borel subgroup of G defined by $B = I(4)$ and U the subgroup of G consisting of matrices that look like $\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$.

- For a quadratic extension L of $\mathbb{Q}$ define
$$GU(n, n) = GU(n, n; L)$$
by
$$GU(n, n)(\mathbb{Q}) = \{g \in GL_{2n}(L) | (\overline{g})^T J_n g = \mu_n(g) J_n, \mu_n(g) \in \mathbb{Q}^\times\}$$
where $\overline{g}$ denotes the conjugate of g.

Denote the algebraic group $GU(2,2;L)$ by $\widetilde{G}$.

- Define
$$\widetilde{\mathbb{H}}_n = \{Z \in M_{2n}(\mathbb{C}) | i(\overline{Z} - Z) \text{ is positive definite}\},$$
$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) | Z = Z^T, i(\overline{Z} - Z) \text{ is positive definite}\}.$$
For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widetilde{G}(\mathbb{R})$, $Z \in \widetilde{H}_2$ define
$$J(g, Z) = CZ + D.$$
The same definition works for $g \in G(\mathbb{R}), Z \in H_2$.

- For v be a finite place of $\mathbb{Q}$, define $L_v = L \otimes_{\mathbb{Q}} \mathbb{Q}_v$.
$\mathbb{Z}_L$ denotes the ring of integers of L and $\mathbb{Z}_{L,v}$ its v-closure in L_v.
Define maximal compact subgroups $\widetilde{K_v}$ and K_v of $\widetilde{G}(\mathbb{Q}_v)$ and $G(\mathbb{Q}_v)$ respectively by
$$\widetilde{K_v} = \widetilde{G}(\mathbb{Q}_v) \cap GL_4(\mathbb{Z}_{L,v}),$$
$$K_v = G(\mathbb{Q}_v) \cap GL_4(\mathbb{Z}_v).$$

- For a positive integer N the subgroups $\Gamma_0(N)$ and $\Gamma^0(N)$ of $SL_2(\mathbb{Z})$ are defined by
$$\Gamma_0(N) = \{A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\}$$
$$\Gamma^0(N) = \{A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod N\}$$
For p a finite place of $\mathbb{Q}$, their local analogues $\Gamma_{0,p}$ (resp. Γ_p^0) are defined by
$$\Gamma_{0,p} = \{A \in GL_2(\mathbb{Z}_p) \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod p\}$$
$$\Gamma_p^0 = \{A \in GL_2(\mathbb{Z}_p) \mid A \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod p\}$$
The local *Iwahori* subgroup I_p is defined to be the subgroup of $K_p = G(\mathbb{Z}_p)$ consisting of those elements of K_p that when reduced mod p lie in the Borel subgroup of $G(\mathbb{F}_p)$. Precisely,
$$I_p = \{A \in K_p \mid A \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \pmod p\}$$

1. Preliminaries

1.1. **Bessel models.** We recall the definition of the Bessel model of Novodvorsky and Piatetski-Shapiro [17] following the exposition of Furusawa [2].

Let $S \in M_2(\mathbb{Q})$ be a symmetric matrix. We let $\mathrm{disc}(S) = -4\det(S)$ and put $d = -\mathrm{disc}(S)$. If $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ then we define the element $\xi = \xi_S = \begin{pmatrix} b/2 & c \\ -a & -b/2 \end{pmatrix}$.

Let L denote the subfield $Q(\sqrt{-d})$ of $\mathbb{C}$.
We always identify $\mathbb{Q}(\xi)$ with L via

$$\mathbb{Q}(\xi) \ni x + y\xi \mapsto x + y\frac{\sqrt{(-d)}}{2} \in L, x, y \in \mathbb{Q}. \tag{1.1.1}$$

We define a subgroup $T = T_S$ of GL_2 by

$$T(\mathbb{Q}) = \{g \in GL_2(\mathbb{Q}) | g^T S g = det(g) S\}. \tag{1.1.2}$$

The center of T is denote by Z_T. It is not hard to verify that $T(\mathbb{Q}) = Q(\xi)^\times$ and $Z_T(\mathbb{Q}) = \mathbb{Q}^\times$. We identify $T(\mathbb{Q})$ with $L^\times$ via (1.1.1)).

We can consider T as a subgroup of G via

$$T \ni g \mapsto \begin{pmatrix} g & 0 \\ 0 & det(g).(g^{-1})^T \end{pmatrix} \in G. \tag{1.1.3}$$

Let us denote by U the subgroup of G defined by

$$U = \{u(X) = \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} | X^T = X\}.$$

Let R be the subgroup of G defined by $R = TU$.

Let ψ be a non trivial character of $\mathbb{A}/\mathbb{Q}$. We define the character $\theta = \theta_S$ on $U(\mathbb{A})$ by $\theta(u(X)) = \psi(tr(S(X)))$. Let Λ be a character of $T(\mathbb{A})/T(\mathbb{Q})$ such that $\Lambda | Z_T(\mathbb{A}^\times) = 1$. Via (1.1.1) we can think of Λ as a character of $L^\times(\mathbb{A})/L^\times$ such that $\Lambda | \mathbb{A}^\times = 1$. Denote by $\Lambda \otimes \theta$ the character of $R(\mathbb{A})$ defined by $(\Lambda \otimes \theta)(tu) = \Lambda(t)\theta(u)$ for $t \in T(\mathbb{A})$ and $u \in U(\mathbb{A})$.

Let π be an automorphic cuspidal representation of $G(\mathbb{A})$ with trivial central character and V_π be its space of automorphic forms.
Then for $\Phi \in V_\pi$, we define a function B_Φ on $G(\mathbb{A})$ by

$$B_\Phi(h) = \int_{R(\mathbb{A})/R(\mathbb{Q})Z_G(\mathbb{A})} (\Lambda \otimes \theta)(r)^{-1} \Phi(rh) dr. \tag{1.1.4}$$

The $\mathbb{C}$ - vector space of function on $\widetilde{G}(\mathbb{A})$ spanned by $\{B_\Phi | \Phi \in V_\pi\}$ is called the global Bessel space of type (S, Λ, ψ) for π. We say that π has a global Bessel model of type (S, Λ, ψ), if the global Bessel space has positive dimension, that is if there exists $\Phi \in V_\pi$ such that $B_\Phi \neq 0$. In Sections 1–7 of this paper, we assume that:
(1.1.5)

There exists S, Λ, ψ such that π has a global Bessel model of type (S, Λ, ψ).

1.2. **Eisenstein series.** We briefly recall the definition of the Eisenstein series used by Furusawa in [2]. Let P be the maximal parabolic subgroup of $\widetilde{G}$ consisting of the elements in $\widetilde{G}$ that look like $\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$. We have the Levi decomposition $P = MN$ with $M = M^{(1)} M^{(2)}$ where the groups $M, N, M^{(1)}, M^{(2)}$ are as defined in [2].

Precisely,

$$M^{(1)}(\mathbb{Q}) = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a \in L^\times \right\} \simeq L^\times. \tag{1.2.1}$$

$$\begin{aligned} M^{(2)}(\mathbb{Q}) &= \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & \lambda & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix} \mid \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GU(1,1)(\mathbb{Q}), \lambda = \mu_1 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\} \\ &\simeq GU(1,1)(\mathbb{Q}). \end{aligned} \tag{1.2.2}$$

$$N(\mathbb{Q}) = \left\{ \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\overline{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a & y \\ 0 & 1 & \overline{y} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q}, x, y \in L \right\}. \tag{1.2.3}$$

We also write

$$m^{(1)}(a) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$m^{(2)} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & \lambda & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}.$$

Let σ be an irreducible automorphic cuspidal representation of $GL_2(\mathbb{A})$ with central character ω_σ. Let χ_0 be a character of $L^\times(\mathbb{A})/L^\times$ such that $\chi_0 \mid \mathbb{A}^\times = \omega_\sigma$.

Finally, let χ be a character of $L^\times(\mathbb{A})/L^\times = M_1(\mathbb{A})M_1(\mathbb{Q})$ defined by

$$\chi(a) = \Lambda(\overline{a})^{-1}\chi_0(\overline{a})^{-1}. \tag{1.2.4}$$

Then defining

$$\Pi(m_1 m_2) = \chi(m_1)(\chi_0 \otimes \sigma)(m_2), m_1 \in M_1(\mathbb{A}), m_2 \in M_2(\mathbb{A}) \tag{1.2.5}$$

we extend σ to an automorphic representation Π of $M(\mathbb{A})$. We regard Π as a representation of $P(\mathbb{A})$ by extending it trivially on $N(\mathbb{A})$. Let δ_P denote the modulus character of P. If $p = m_1 m_2 n \in P(\mathbb{A})$ with $m_i \in M_i(\mathbb{A})(i = 1, 2)$ and $n \in N(\mathbb{A})$,

$$\delta_P(p) = |N_{L/\mathbb{Q}}(m_1)|^3 \cdot |\mu_1(m_2)|^{-3}, \tag{1.2.6}$$

where $||$ denoted the modulus function on $\mathbb{A}$.

Then for $s \in \mathbb{C}$, we form the family of induced automorphic representations of $\widetilde{G}(\mathbb{A})$

$$I(\Pi, s) = \mathrm{Ind}_{P(\mathbb{A})}^{\widetilde{G}(\mathbb{A})}(\Pi \otimes \delta_P^s) \tag{1.2.7}$$

where the induction is normalized. Let $f(g, s)$ be an entire section in $I(\Pi, s)$ viewed concretely as a complex-valued function on $\widetilde{G}(\mathbb{A})$ which is left $N(\mathbb{A})$-invariant and such that for each fixed $g \in \widetilde{G}(\mathbb{A})$, the function $m \mapsto f(mg, s)$ is a cusp form on

$M(\mathbb{A})$ for the automorphic representation $\Pi \otimes \delta_P^s$. Finally we form the Eisenstein series $E(g,s) = E(g,s;f)$ by

$$E(g,s) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \widetilde{G}(\mathbb{Q})} f(\gamma g, s) \tag{1.2.8}$$

for $g \in \widetilde{G}(\mathbb{A})$.

This series converges absolutely (and uniformly in compact subsets) for $\mathrm{Re}(s) > 1/2$, has a meromorphic extension to the entire plane and satisfies a functional equation (see [14, 2]).

2. The Rankin-Selberg integral

2.1. **The global integral.** The main object of study in this paper is the following global integral of Rankin-Selberg type

$$Z(s) = Z(s,f,\Phi) = \int_{G(\mathbb{Q})Z_G(\mathbb{A})\backslash G(\mathbb{A})} E(g,s,f)\Phi(g)dg, \tag{2.1.1}$$

where $\Phi \in V_\pi$ and $f \in I(\Pi, s)$. $Z(s)$ converges absolutely away from the poles of the Eisenstein series.

Let $\Theta = \Theta_S$ be the following element of $\widetilde{G}(\mathbb{Q})$

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & -\overline{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ where } \alpha = \frac{b + \sqrt{-d}}{2c}$$

The 'basic identity' proved by Furusawa in [2] is that

$$Z(s) = \int_{R(\mathbb{A})\backslash G(\mathbb{A})} W_f(\Theta h, s) B_\Phi(h) dh \tag{2.1.2}$$

where for $g \in \widetilde{G}(\mathbb{A})$ we have

$$W_f(g,s) = \int_{\mathbb{A}/\mathbb{Q}} f\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g, s\right) \psi(cx) dx. \tag{2.1.3}$$

and B_Φ is the Bessel model of type (S, Λ, ψ) defined in section 1.

2.2. **The local integral.** In this section v refers to any place of $\mathbb{Q}$. Let $\pi = \otimes_v \pi_v$ and $\sigma = \otimes_v \sigma_v$. Now suppose that Φ and f are factorizable functions with $\Phi = \otimes_v \Phi_v$ and $f(\ ,s) = \otimes_v f_v(\ ,s)$.

By the uniqueness of the Whittaker and the Bessel models, we have

$$W_f(g,s) = \prod_v W_{f,v}(g_v, s) \tag{2.2.1}$$

$$B_\Phi(h) = \prod_v B_{\Phi,v}(h_v, s) \tag{2.2.2}$$

for $g = (g_v) \in \widetilde{G}(\mathbb{A})$ and $h = (h_v) \in G(\mathbb{A})$ and *local* Whittaker and Bessel functions $W_{f,v}$, $B_{\Phi,v}$ respectively. Henceforth we write $W_v = W_{f,v}, B_v = B_{\Phi,v}$ when no confusion can arise.

Therefore our global integral breaks up as a product of local integrals

$$Z(s) = \prod_v Z_v(s) \tag{2.2.3}$$

where

$$Z_v(s) = Z_v(s, W_v, B_v) = \int_{R(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)} W_v(\Theta g, s) B_v(g) dg.$$

2.3. **The unramified case.** The local integral is evaluated in [2] in the unramified case. We recall the result here.

Suppose that the characters $\omega_\pi, \omega_\sigma, \chi_0$ are trivial. Now let q be a finite prime of $\mathbb{Q}$ such that

(a) The local components π_q, σ_q and Λ_q are all unramified.
(b) The conductor of ψ_q is $\mathbb{Z}_q$.
(c) $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_q)$ with $c \in \mathbb{Z}_q^\times$.
(d) $-d = b^2 - 4ac$ generates the discriminant of $L_q/\mathbb{Q}_q$.

Since σ_q is spherical, it is the spherical principal series representation induced from unramified characters α_q, β_q of $\mathbb{Q}_q^\times$.

Suppose M_0 is the maximal torus (the group of diagonal matrices) inside G and P_0 the Borel subgroup containing M_0 as Levi component. π_q is a spherical principal series representation, so there exists an unramified character γ_q of $M_0(\mathbb{Q}_q)$ such that $\pi_q = Ind_{P_0(\mathbb{Q}_q)}^{M_0(\mathbb{Q}_q)} \gamma_q$, (where we extend γ_q to P_0 trivially). We define characters $\gamma_q^{(i)} (i = 1, 2, 3, 4)$ of $\mathbb{Q}_q^\times$ by

$$\gamma_q^{(1)}(x) = \gamma_q \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma_q^{(2)}(x) = \gamma_q \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix},$$

$$\gamma_q^{(3)}(x) = \gamma_q \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}, \qquad \gamma_q^{(4)}(x) = \gamma_q \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now let $f_q(\ ,s)$ be the unique normalized $\widetilde{K_q}-$ spherical vector in $I_q(\Pi_q, s)$ and Φ_q be the unique normalized K_q- spherical vector in π_q. Let W_q, B_q be the coresponding vectors in the local Whittaker and Bessel spaces. The following result is proved in [2]

Theorem 2.3.1 (Furusawa)**.** *Let $\rho(\Lambda_q)$ denote the Weil representation of $GL_2(\mathbb{Q}_q)$ corresponding to Λ_q. Then we have*

$$Z_q(s, W_q, B_q) = \frac{L(3s + \frac{1}{2}, \pi_q \times \sigma_q)}{L(6s + 1, \mathbf{1}) L(3s + 1, \sigma_q \times \rho(\Lambda_q))}$$

where,

$$L(s, \pi_q \times \sigma_q) = \prod_{i=1}^{4} \left((1 - \gamma_q^{(i)} \alpha_q(q) q^{-s})(1 - \beta_q^{(i)} \alpha_q(q) q^{-s}) \right)^{-1},$$
$$L(s, \mathbf{1}) = (1 - q^{-s})^{-1},$$

$$L(s, \sigma_q \times \rho(\Lambda_q))$$
$$= \begin{cases} (1-\alpha_q^2(q)q^{-2s})^{-1}(1-\beta_q^2(q)q^{-2s})^{-1} & \text{if } q \text{ is inert in } L, \\ \\ (1-\alpha_q(q)\Lambda_q(q_1)q^{-s})^{-1}(1-\beta_q(q)\Lambda_q(q_1)q^{-s})^{-1} & \text{if } q \text{ is ramified in } L, \\ \\ (1-\alpha_q(q)\Lambda_q(q_1)q^{-s})^{-1}(1-\beta_q(q)\Lambda_q(q_1)q^{-s})^{-1} & \\ \cdot(1-\alpha_q(q)\Lambda_q^{-1}(q_1)q^{-s})^{-1}(1-\beta_q(q)\Lambda_q^{-1}(q_1)q^{-s})^{-1} & \text{if } q \text{ splits in } L, \end{cases}$$

where $q_1 \in \mathbb{Z}_q \otimes_{\mathbb{Q}} L$ *is any element with* $N_{L/\mathbb{Q}}(q_1) \in q\mathbb{Z}_q^{\times}$.

3. Strategy for computing the p-adic integral

3.1. **Assumptions.** Throughout this section we fix an odd prime p in $\mathbb{Q}$ such that *p is inert in L*. Moreover, we assume that $S \in M_2(\mathbb{Z}_p)$.

The fact that p is inert in L implies that if w, z are elements of $\mathbb{Z}_p$ then $w + z\xi \in (T(\mathbb{Q}_p) \cap K_p)$ if and only if at least one of w, z is an unit.

Moreover the additional assumption $S \in M_2(\mathbb{Z}_p)$ forces that a, c are units in $\mathbb{Z}_p$.

3.2. **An explicit set of coset representatives.** Recall the Iwahori subgroup I_p. It will be useful to describe a set of coset representatives of K_p/I_p.

But first some definitions.

Let Y be the set $\{0, 1, .., p-1\}$. Let $V = Y \cup \{\infty\}$ where ∞ is just a convenient formal symbol.

For $x = (n, q, r) \in \mathbb{Z}_p^3$, let $U_x \in U(\mathbb{Q}_p)$ be the matrix $\begin{pmatrix} 1 & 0 & n & q \\ 0 & 1 & q & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

For $y \in \mathbb{Z}_p$ define $Z_y = \begin{pmatrix} 1 & y & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -y & 1 \end{pmatrix} \in K_p$.

Also define $Z_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in K_p$.

In particular, the definitions U_x, Z_y make sense for $x \in Y^3, y \in V$. Now we define the following three classes of matrices. We call them matrices of class A, class B and class D respectively.

(a) For $x = (n, q, r) \in Y^3, y \in V$, let $A_x^y = U_x J Z_y$.
(b) For $x = (n, q, r) \in Y^3$ with $q^2 - nr \equiv 0 \pmod p$ and $y \in V$, let $B_x^y = J U_x J Z_y$.

(c) For $\lambda, y \in V$, let
$$D_\lambda^y = \begin{cases} \begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 1 & 0 & 0 & \lambda^{-1} \\ 0 & 1 & \lambda^{-1} & 0 \\ 0 & \lambda & -1 & 0 \end{pmatrix} Z_y & \text{if } \lambda \neq 0, \infty, \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} Z_y & \text{if } \lambda = 0, \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} Z_y & \text{if } \lambda = \infty. \end{cases}$$

Let S be the set obtained obtained by taking the union of the class A, class B and class D matrices, precisely $S = \{A_x^y\}_{\substack{y\in V \\ x \in Y^3}} \bigcup \{B_x^y\}_{\substack{y \in V, x=(n,q,r)\in Y^3 \\ q^2-nr\equiv 0 \pmod p}} \bigcup \{D_\lambda^y\}_{\substack{\lambda\in V \\ y \in V}}$. Clearly S has cardinality $p^3(p+1) + p^2(p+1) + (p+1)^2 = (p+1)^2(p^2+1)$.

Lemma 3.2.1. *S is a complete set of coset representatives for K_p/I_p*

Proof. Let us first verify that S has the *right* cardinality. Clearly the cardinality of K_p/I_p is the same as the cardinality of $G(\mathbb{F}_p)/B(\mathbb{F}_p)$ where B is the Borel subgroup of G. By [13, Theorem 3.2], $|G(\mathbb{F}_p)| = p^4(p-1)^3(p+1)^2(p^2+1)$. On the other hand B has the Levi-decomposition

$$B = \begin{pmatrix} g & 0 \\ 0 & v.(g^{-1})^T \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix}$$

with g upper-triangular, X symmetric and $v \in GL(1)$. So $|B(\mathbb{F}_p)| = p^4(p-1)^3$. Thus $|G(\mathbb{F}_p)/B(\mathbb{F}_p)| = (p+1)^2(p^2+1)$ which is the same as the cardinality of S.

So it is enough to show that no two matrices in S lie in the same coset.

For a 2×2 matrix H with coefficients in $\mathbb{Z}_p$, we may reduce H mod p and consider the $\mathbb{F}_p$-rank of the resulting matrix; we denote this quantity by $r_p(H)$. It is easy to see that if the matrix $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ varies in a fixed coset of K_p/I_p, the pair $(r_p(A_1), r_p(A_3))$ remains constant.

Observe now that if A is of class A, then $r_p(A_3) = 2$; for A of class B, $r_p(A_3) < 2$ and $r_p(A_1) = 2$; while for A of class D we have $r_p(A_3) < 2$, $r_p(A_1) < 2$. This proves that elements of S of different classes cannot lie in the same coset.

Now we consider distinct elements of S of the same class, and show that they too must lie in different cosets.

For $x_1 = (n_1, q_1, r_1), x_2 = (n_2, q_2, r_2) \in Y^3$, $y_1, y_2 \in Y$, consider the elements $A_{x_1}^{y_1}, A_{x_2}^{y_2}, B_{x_1}^{y_1}, B_{x_2}^{y_2}$ of S. We have

$(A_{x_1}^{y_1})^{-1} A_{x_2}^{y_2} = (B_{x_1}^{y_1})^{-1} B_{x_2}^{y_2} =$

$$\begin{pmatrix} 1 & y_2 - y_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -n_2 + n_1 & -n_2y_2 - q_2 + n_1y_2 + q_1 & 1 & 0 \\ y_1(n_1 - n_2) + q_1 - q_2 & y_1y_2(n_1 - n_2) + (y_1 + y_2)(q_1 - q_2) - r_2 + r_1 & y_1 - y_2 & 1 \end{pmatrix}.$$

So if the above matrix belongs to I_p, we must have $y_1 = y_2, n_1 = n_2$. That leads to $q_1 = q_2$, and finally by looking at the bottom row we conclude $r_1 = r_2$.

This covers the case of class A and class B matrices in S whose y-component is not equal to ∞.

Now $(A_{x_1}^{y_1})^{-1} A_{x_2}^{\infty} = (B_{x_1}^{y_1})^{-1} B_{x_2}^{\infty} =$

$$\begin{pmatrix} -y_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ q_1 - q_2 & n_1 - n_2 & 0 & 1 \\ q_1y_1 + r_1 - y_1q_2 - r_2 & n_1y_1 + q_1 - y_1n_2 - q_2 & 1 & y_1 \end{pmatrix}$$

which cannot belong to I_p.

Also $(A_{x_1}^{\infty})^{-1} A_{x_2}^{\infty} = (B_{x_1}^{\infty})^{-1} B_{x_2}^{\infty} =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -r_2 + r_1 & -q_2 + q_1 & 1 & 0 \\ -q_2 + q_1 & -n_2 + n_1 & 0 & 1 \end{pmatrix}$$

and if the above matrix lies in I_p we must have $x_1 = x_2$.

Thus we have completed the proof for class A and class B matrices. We now consider the class D matrices.

Let $m_1, m_2 \in Y \setminus \{0\}$.

We have $(D_{m_1}^{y_1})^{-1} D_{m_2}^{y_2} =$

$$\frac{1}{2}\begin{pmatrix} \frac{m_2}{m_1} + 1 & (\frac{m_2}{m_1} + 1)(y_2 - y_1) & (\frac{1}{m_1} - \frac{1}{m_2})(y_2 + y_1) & -m_1 + m_2 \\ 0 & \frac{m_2}{m_1} + 1 & m_1 + m_2 & 0 \\ 0 & m_1 - m_2 & 1 + \frac{m_1}{m_2} & 0 \\ m_1 - m_2 & (y_1 + y_2)(m_1 - m_2) & (\frac{m_1}{m_2} + 1)(y_1 - y_2) & \frac{m_1}{m_2} + 1 \end{pmatrix}$$

If the above matrix belongs to I_p then we must have $m_1 = m_2$ which implies that $y_2 = y_1$.

$(D_{m_1}^{y_1})^{-1} D_0^{y_2} =$

$$\frac{1}{2}\begin{pmatrix} -1 & -y_2 + y_1 & \frac{y_2 - y_1}{m_1} & \frac{-1}{m_1} \\ 0 & 1 & \frac{1}{m_1} & 0 \\ 0 & m_1 & -1 & 0 \\ -m_1 & -m_1y_2 + m_1y_1 & -y_2 - y_1 & 1 \end{pmatrix}$$

which, if in I_p, implies that $m_1 = 0$, a contradiction.

$(D_{m_1}^{y_1})^{-1} D_{\infty}^{y_2} =$

$$\frac{1}{2}\begin{pmatrix} \frac{1}{m_1} & \frac{y_2 - y_1}{m_1} & -y_2 - y_1 & 1 \\ 0 & \frac{1}{m_1} & 1 & 0 \\ 0 & -1 & m_1 & 0 \\ -1 & -y_2 - y_1 & -m_1y_2 + m_1y_1 & m_1 \end{pmatrix}$$

which cannot belong to I_p.
$(D_0^{y_1})^{-1}D_\infty^{y_2} =$

$$\begin{pmatrix} 0 & 0 & y_2 - y_1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -y_2 + y_1 & 0 & 0 \end{pmatrix}$$

which cannot belong to I_p.
Next, $(D_{m_1}^{y_1})^{-1}D_{m_2}^{\infty} =$

$$\frac{1}{2}\begin{pmatrix} -y_1(1+\frac{m_2}{m_1}) & \frac{m_2}{m_1}+1 & -\frac{1}{m_1}+\frac{1}{m_2} & y_1 - \frac{y_1}{m_2} - \frac{1}{m_1} \\ \frac{m_2}{m_1}+1 & 0 & 0 & -\frac{1}{m_1}+\frac{1}{m_2} \\ m_1 - m_2 & 0 & 0 & 1+\frac{m_1}{m_2} \\ m_1y_1 - y_1m_2 & m_1 - m_2 & 1+\frac{m_1}{m_2} & (1+\frac{m_1}{m_2})y_1 \end{pmatrix}$$

If the above matrix belongs to I_p we must simultaneously have $m_1 - m_2 = 0$ and $m_1/m_2 = -1$, which is not possible.
$(D_{m_1}^{y_1})^{-1}D_0^{\infty} =$

$$\frac{1}{2}\begin{pmatrix} -y_1 & -1 & -\frac{1}{m_1} & -\frac{y_1}{m_1} \\ 1 & 0 & 0 & \frac{1}{m_1} \\ m_1 & 0 & 0 & -1 \\ m_1y_1 & -m_1 & 1 & -y_1 \end{pmatrix}$$

which cannot belong to I_p.
$(D_{m_1}^{y_1})^{-1}D_\infty^{\infty} =$

$$\frac{1}{2}\begin{pmatrix} -\frac{y_1}{m_1} & \frac{1}{m_1} & 1 & -y_1 \\ \frac{1}{m_1} & 0 & 0 & 1 \\ -1 & 0 & 0 & m_1 \\ -y_1 & -1 & m_1 & m_1y_1 \end{pmatrix}$$

which cannot belong to I_p.
$(D_0^{y_1})^{-1}D_\infty^{\infty} =$

$$\begin{pmatrix} 0 & 0 & -1 & -y_2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ y_2 & -1 & 0 & 0 \end{pmatrix}$$

which cannot belong to I_p.
$(D_{m_1}^{\infty})^{-1}D_{m_2}^{\infty} =$

$$\frac{1}{2}\begin{pmatrix} \frac{m_1+m_2}{m_1} & 0 & 0 & \frac{m_1-m_2}{m_2m_1} \\ 0 & \frac{m_1+m_2}{m_1} & \frac{m_1-m_2}{m_2m_1} & 0 \\ 0 & m_1 - m_2 & \frac{m_1+m_2}{m_2} & 0 \\ m_1 - m_2 & 0 & 0 & \frac{m_1+m_2}{m_2} \end{pmatrix}$$

which cannot belong to I_p unless $m_1 = m_2$.
$(D_{m_1}^{\infty})^{-1}D_0^{\infty} =$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \frac{1}{m_1} \\ 0 & -1 & -\frac{1}{m_1} & 0 \\ 0 & -m_1 & 1 & 0 \\ m_1 & 0 & 0 & -1 \end{pmatrix}$$

which, if in I_p implies $m_1 = 0$ a contradiction.

$(D^{\infty}_{m_1})^{-1}D^{\infty}_{\infty} =$

$$\begin{pmatrix} \frac{1}{m_1} & 0 & 0 & 1 \\ 0 & \frac{1}{m_1} & 1 & 0 \\ 0 & -1 & m_1 & 0 \\ -1 & 0 & 0 & m_1 \end{pmatrix}$$

which cannot belong to I_p.

$(D^{\infty}_{0})^{-1}D^{\infty}_{\infty} =$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which does not belong to I_p.

Finally, observe that for any $y_3, y_4 \in V$, we have $(D^{y_3}_0)^{-1}D^{y_4}_0 = (D^{y_3}_{\infty})^{-1}D^{y_4}_{\infty} = Z_{y_3-y_4}$ which does not belong to I_p if $y_3 \neq y_4$. (Here we interpret $y_3 - y_4 \pmod p$) and we also define $\infty - \infty = 0$ and $y_3 - y_4 = \infty$ if one of them equals ∞).

Thus we have covered all cases and this completes the proof. □

3.3. **Reducing the integral to a sum.** By [2, p. 201]) we have the following disjoint union

$$G(\mathbb{Q}_p) = \coprod_{\substack{l\in\mathbb{Z} \\ 0\le m\in\mathbb{Z}}} R(\mathbb{Q}_p)\cdot h(l,m)\cdot K_p \tag{3.3.1}$$

where

$$h(l,m) = \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & p^{m+l} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p^m \end{pmatrix}.$$

We wish to compute

$$Z_p(s) = \int_{R(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} W_p(\Theta h, s) B_p(h) dh. \tag{3.3.2}$$

By (3.3.1) and (3.3.2) we have

$$Z_p(s) = \sum_{l\in\mathbb{Z}, m\ge 0} \int_{R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(l,m)K_p} W_p(\Theta h, s) B_p(h) dh. \tag{3.3.3}$$

For $m \geq 0$ we define the subset T_m of S by

$T_m = \{B^0_{(1,0,0)}, B^{\infty}_{(1,0,0)}, B^0_{(0,0,1)}, B^{\infty}_{(0,0,1)}, B^0_{(0,0,0)}, B^{\infty}_{(0,0,0)}, A^0_{(0,0,0)}, A^{\infty}_{(0,0,0)}\}$
if $m > 0$,

$T_0 = \{B^0_{(1,0,0)}, B^{\infty}_{(1,0,0)}, B^0_{(0,0,0)}, A^0_{(0,0,0)}\}$.

Also, we use the notation $t_1 = B^0_{(1,0,0)}, t_2 = B^{\infty}_{(1,0,0)}, ..., t_8 = A^{\infty}_{(0,0,0)}$. Thus $T_m = \{t_i | 1 \leq i \leq 8\}$ if $m > 0$ and $T_0 = \{t_1, t_2, t_5, t_7\}$.

Proposition 3.3.1. *Let $l \in \mathbb{Z}, m \geq 0$. Then we have*

$$R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(l,m)K_p = \coprod_{t\in T_m} R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(l,m)tI_p$$

Proof. Define two elements f and g in K_p to be (l, m)-equivalent if there exists $r \in R(\mathbb{Q}_p)$ and $k \in I_p$ such that $rh(l, m)fk = h(l, m)g$. Furthermore observe that if two elements of K_p are congruent $\mod p$ then they are in the same I_p-coset and therefore are trivially (l, m)-equivalent.

The proposition can be restated as saying that any $s \in S$ is (l, m)-equivalent to exactly one of the elements t with $t \in T_m$. This will follow from the following nine claims which we prove later below.

Claim 1. *Any class A matrix in S by left-multiplying by an appropriate element of $U(\mathbb{Z}_p)$ can be made congruent $\mod p$ to $A^y_{(0,0,0)}$ for some $y \in V$.*

Claim 2. *If $m > 0$ all the $A^y_{(0,0,0)}, y \in V \setminus \{0\}$ are (l, m)-equivalent. In the case $m = 0$ all the $A^y_{(0,0,0)}, y \in V$ are $(l, 0)$-equivalent.*

Claim 3. *Any class B matrix in S by left-multiplying by an appropriate element of $U(\mathbb{Z}_p)$ can be made congruent $\mod p$ to one of the matrices*

$$B^{-\lambda}_{(1,\lambda,\lambda^2)}, B^{\infty}_{(1,\lambda,\lambda^2)} B^y_{(0,0,1)}, B^y_{(0,0,0)},$$

where $\lambda \in Y, y \in V$.

Claim 4. *The matrices $B^y_{(1,\lambda,\lambda^2)}, \lambda \in Y, y \in \{-\lambda, \infty\}$ are all (l, m)-equivalent to one of the matrices $B^y_{(1,0,0)}, y \in \{0, \infty\}$.*

Claim 5. *The matrices $B^y_{(0,0,1)}, y \in V$ by left-multiplying by an appropriate element of $U(\mathbb{Z}_p)$ can be made equal to one of the matrices $B^y_{(0,0,1)}$ with $y \in \{0, \infty\}$.*

Claim 6. *The matrices $B^y_{(0,0,0)}, y \in V$ are (l, m)-equivalent to one of the matrices $B^y_{(0,0,0)}$ with $y \in \{0, \infty\}$. In the case $m = 0$ these two matrices are also equivalent.*

Claim 7. *The matrices $B^0_{(1,0,0)}, B^{\infty}_{(0,0,1)}$ are $(l, 0)$-equivalent and the matrices $B^{\infty}_{(1,0,0)}, B^0_{(0,0,1)}$ are also $(l, 0)$-equivalent.*

Claim 8. *Any class D matrix D^y_{λ} by left-multiplying by an appropriate element of $U(\mathbb{Z}_p)$ can be made equal to a class B matrix.*

Claim 9. *No two elements of T_m are (l, m)-equivalent for any $m \geq 0$.*

Indeed claims 1, 2 imply that any class A matrix is (l, m)-equivalent to one of t_7, t_8 (and when $m = 0$, t_7 alone suffices). On the other hand claims 3,4,5,6,7 tell us that any class B matrix is (l, m)-equivalent to one of the $t_i, 1 \leq i \leq 6$ (and that just t_1, t_2, t_5 suffice if $m = 0$). Also claim 8 says that any class D matrix is also (l, m)-equivalent to one of the above. Since the class A, class B and class D matrix exhaust S, this shows that any element of S is (l, m)-equivalent to some element of T_m; in other words we do have the union stated in Proposition 3.3.1. Finally claim 9 completes the argument by implying that the union is indeed disjoint.

□

We now prove each of the above claims. The proofs are just computations, we simply multiply by suitable elements of R to get the results we desire.

Proof of claim 1. This follows from the fact that $U_{-x} A^y_x \equiv JZ_y \pmod p$ and $JZ_y = A^y_{(0,0,0)}$. □

Proof of claim 2. We first deal with the case $m = 0$. For $y \in V, y \neq 0$ let $j = (-\frac{a}{y} + \frac{b}{2}) + \xi \in (T(\mathbb{Q}_p) \cap K_p)$ (here and elsewhere we interpret $1/\infty = 0$). Consider the element $(A^0_{(0,0,0)})^{-1} h(l,0)^{-1} j h(l,0) A^y_{(0,0,0)}$. By direct calculation this equals

$$\begin{pmatrix} -\frac{a}{y} & 0 & 0 & 0 \\ -c & \frac{-cy^2+a-yb}{y} & 0 & 0 \\ 0 & 0 & -\frac{cy^2+a-yb}{y} & c \\ 0 & 0 & 0 & -\frac{a}{y} \end{pmatrix}$$

if $y \neq \infty$ and equals

$$\begin{pmatrix} a & 0 & 0 & 0 \\ b & -c & 0 & 0 \\ 0 & 0 & c & b \\ 0 & 0 & 0 & -a \end{pmatrix}$$

if $y = \infty$. Both these matrices lie in I_p and this proves the claim for $m = 0$.

Now consider $m > 0$. For $y \in V, y \neq 0, \infty$, let $j = cy + p^m\xi \in (T(\mathbb{Q}_p) \cap K_p)$. Consider the element $(A^\infty_{(0,0,0)})^{-1}\ h(l,m)^{-1} j\ h(l,m) A^y_{(0,0,0)}$, which by direct calculation equals

$$\begin{pmatrix} -c & \frac{bp^m}{2} & 0 & 0 \\ cy - \frac{bp^m}{2} & cy^2 - \frac{yp^m b}{2} + p^{2m}a & 0 & 0 \\ 0 & 0 & -p^{2m}a - cy^2 + \frac{yp^m b}{2} & cy - \frac{bp^m}{2} \\ 0 & 0 & \frac{bp^m}{2} & c \end{pmatrix}$$

and this lies in I_p. Thus $A^y_{0,0,0}$ is (l,m)-equivalent to $A^\infty_{0,0,0}$ and this completes the proof of the claim. □

Proof of claim 3. Before proving this claim let us make a small remark. If $\lambda \in Y$ is such that λ^2 does not belong to Y one may ask what we mean by the notation $B^y_{(1,\lambda,\lambda^2)}$; in such a case, we understand λ^2 to refer to the unique element in Y that is congruent to $\lambda^2 \mod p$. *This convention will govern any such situation.*

We now begin proving the claim. Given a class B matrix $B^y_{(n,q,r)}$ with $n \neq 0$ we must have $q \equiv n\lambda, r \equiv n\lambda^2 \pmod p$ for some appropriate $\lambda \in Y$.

First assume that $y \neq -\lambda$. If $y \neq \infty$ put $s = \frac{\lambda - y + n(\lambda+y)}{n(y+\lambda)}, t = -\frac{1}{n(y+\lambda)}, u = 0$ and check that $(B^y_{n,q,r})^{-1} U_{s,t,u} B^\infty_{1,\lambda,\lambda^2}$ is congruent $\mod p$ to

$$\begin{pmatrix} -\lambda - y & 0 & -\frac{1}{n(y+\lambda)} & \frac{\lambda+n(y+\lambda)}{n(y+\lambda)} \\ \frac{\lambda+n(y+\lambda)}{n(y+\lambda)} & \frac{1}{n(y+\lambda)} & 0 & -\frac{1}{n(y+\lambda)} \\ 0 & 0 & -\frac{1}{y+\lambda} & \frac{\lambda+n(y+\lambda)}{y+\lambda} \\ 0 & 0 & 0 & n(y+\lambda) \end{pmatrix}.$$

If $y = \infty$ put $s = \frac{n-1}{n}, t = 0, u = 0$ and check that $(B^\infty_{n,q,r})^{-1} U_{s,t,u} B^\infty_{1,\lambda,\lambda^2}$ is congruent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{(n-1)\lambda}{n} & \frac{1}{n} & 0 & \frac{n-1}{n} \\ 0 & 0 & 1 & (n-1)\lambda \\ 0 & 0 & 0 & n \end{pmatrix}.$$

Both these matrices belong to I_p.

Now suppose that $y = -\lambda$. Put $s = \frac{n-1}{n}, t = 0, u = 0$ and observe that $(B^{-\lambda}_{(n,q,r)})^{-1}U_{(s,t,u)}B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is congruent $\mod p$ to

$$\begin{pmatrix} \frac{1}{n} & 0 & \frac{n-1}{n} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which belongs to I_p also.

Finally assume that $n = 0$. So $q = 0$ as well. If $r = 0$ there is nothing to prove. So suppose $r \neq 0$. If $y \neq \infty$ put $s = 0, t = \frac{y(r-1)}{r}, u = \frac{r-1}{r}$ and observe that $(B^{y}_{(0,0,r)})^{-1}U_{(s,t,u)}B^{y}_{(0,0,1)}$ equals

$$\begin{pmatrix} 1 & 0 & -\frac{(r-1)y^2)}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix}$$

which belongs to I_p. If $y = \infty$ put $s = 0, t = 0, u = \frac{r-1}{r}$ and observe that $(B^{y}_{(0,0,r)})^{-1}U_{(s,t,u)}B^{y}_{(0,0,1)}$ equals

$$\begin{pmatrix} \frac{1}{r} & 0 & \frac{r-1}{r} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which belongs to I_p.

Thus the claim is proved. □

Proof of claim 4. First suppose that $y = \infty$. If $m > 0$, put $j = c/\lambda + p^m\xi \in (T(\mathbb{Q}_p) \cap K_p)$. By direct calculation we verify that $(B^{\infty}_{(1,0,0)})^{-1}h(l,m)^{-1}jh(l,m)B^{\infty}_{(1,\lambda,\lambda^2)}$ is congruent mod$p$ to

$$\begin{pmatrix} \frac{c}{\lambda} & 0 & 0 & 0 \\ c & \frac{c}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{c}{\lambda} & -c \\ 0 & 0 & 0 & \frac{c}{\lambda} \end{pmatrix}$$

and this belongs to I_p.

If $m = 0$, put $j = \frac{(2c-b\lambda)}{2\lambda} + \xi \in (T(\mathbb{Q}_p) \cap K_p)$ and check that if we let $n \in Y \setminus 0$ be the element congruent $\mod p$ to $\frac{a\lambda^2 - b\lambda + c}{c}$ and $y \in Y \setminus 0$ be the element congruent $\mod p$ to $-\frac{c}{a\lambda}$ then $(B^{y}_{(n,0,0)})^{-1}h(l,0)^{-1}jh(l,0)B^{\infty}_{(1,\lambda,\lambda^2)}$ is congruent mod$p$ to

$$\begin{pmatrix} \frac{c(a\lambda^2-b\lambda+c)}{a\lambda^2} & 0 & 0 & 0 \\ \frac{c-b\lambda}{\lambda} & -a & 0 & 0 \\ 0 & 0 & a & \frac{c-b\lambda}{\lambda} \\ 0 & 0 & 0 & -\frac{c(a\lambda^2-b\lambda+c)}{a\lambda^2} \end{pmatrix}$$

which lies in I_p.Hence $B^{\infty}_{(1,\lambda,\lambda^2)}$ is $(l,0)$-equivalent to $B^{y}_{(n,0,0)}$ and by the proof of Claim 3 it follows that it is $(l,0)$-equivalent to $B^{\infty}_{(1,0,0)}$.

Now assume that $y = -\lambda$. If $\lambda = 0$ there is nothing to prove so assume $\lambda \neq 0$. If $m > 0$, put $j = c + p^m\lambda\xi \in (T(\mathbb{Q}_p) \cap K_p)$. By a direct calculation we see that $(B^{0}_{(1,0,0)})^{-1}h(l,m)^{-1}jh(l,m)B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is congruent to $cI_4 \pmod p$ and thus $B^{-\lambda}_{(1,\lambda,\lambda^2)}$

is (l,m)-equivalent to $B^0_{(1,0,0)}$. If $m=0$, put $j = \frac{(2c-b\lambda)}{2\lambda} + \xi \in (T(\mathbb{Q}_p) \cap K_p)$ and check that if we let $n \in Y \setminus 0$ be the element that is congruent mod$p$ to $\frac{a\lambda^2 - b\lambda + c}{c}$ then $(B^0_{(n,0,0)})^{-1} h(l,0)^{-1} j h(l,0) B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is congruent $\mod p$ to

$$\begin{pmatrix} \frac{c}{\lambda} & 0 & 0 & 0 \\ -a & \frac{a\lambda^2 - b\lambda + c}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{a\lambda^2 - b\lambda + c}{\lambda} & a \\ 0 & 0 & 0 & \frac{c}{\lambda} \end{pmatrix}$$

which lies in I_p. Hence $B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is $(l,0)$-equivalent to $B^0_{(n,0,0)}$ and by Claim 3 it follows that it is $(l,0)$-equivalent to $B^0_{(1,0,0)}$. □

Proof of claim 5. If $y = \infty$ there is nothing to prove. So assume $y \in Y$. Put $s=0, t=-y, u=0$ and observe that $(B^y_{(0,0,1)})^{-1} U_{(s,t,u)} B^0_{(0,0,1)}$ equals

$$\begin{pmatrix} 1 & 0 & y^2 & -y \\ 0 & 1 & -y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is in I_p. □

Proof of claim 6. First consider the case $m > 0$. Take $y \in V \setminus \{0, \infty\}$ and let $j = c/y + p^m \xi \in (T(\mathbb{Q}_p) \cap K_p)$. By direct calculation we verify that

$$(B^y_{(0,0,0)})^{-1} h(l,m)^{-1} j h(l,m) B^0_{(0,0,0)}$$

is congruent mod$p$ to $\frac{c}{y} I_4$ and so $B^y_{(0,0,0)}$ is (l,m)-equivalent to $B^0_{(0,0,0)}$.

Now let $m = 0$. Take $y \in V \setminus \{0, \infty\}$ and let $j = c/y + b/2 + \xi \in (T(\mathbb{Q}_p) \cap K_p)$. By direct calculation we verify that $(B^y_{(0,0,0)})^{-1} h(l,0)^{-1} j h(l,0) B^0_{(0,0,0)}$ equals

$$\begin{pmatrix} \frac{ay^2+by+c}{y} & 0 & 0 & 0 \\ -a & \frac{c}{y} & 0 & 0 \\ 0 & 0 & \frac{c}{y} & a \\ 0 & 0 & 0 & \frac{ay^2+by+c}{y} \end{pmatrix}$$

which lies in I_p.

Finally, if we take $j = b/2 + \xi \in (T(\mathbb{Q}_p) \cap K_p)$ we can verify that

$$(B^\infty_{(0,0,0)})^{-1} h(l,0)^{-1} j h(l,0) B^0_{(0,0,0)}$$

$$= \begin{pmatrix} -a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & -c & b \\ 0 & 0 & 0 & a \end{pmatrix}$$

which lies in I_p. □

Proof of claim 7. Putting $j = \frac{b}{2} + \xi$ and $s = 1 - \frac{c}{a}, t = 0, u = 0$ we verify that

$$(B^\infty_{(0,0,1)})^{-1} h(l,0)^{-1} j U_{(s,t,u)} h(l,0) B^0_{(1,0,0)}$$

$$= \begin{pmatrix} -c & 0 & c-a & 0 \\ \frac{bc}{a} & c & \frac{b(a-c)}{a} & 0 \\ 0 & 0 & -a & b \\ 0 & 0 & 0 & a \end{pmatrix}$$

which lies in I_p.

Putting $j = -\frac{b}{2} + \xi$ and $u = 1 - \frac{a}{c}, t = 0, s = 0$ we verify that

$$(B^{\infty}_{(0,0,1)})^{-1} h(l,0)^{-1} j U_{(s,t,u)} h(l,0) B^{0}_{(1,0,0)}$$

$$= \begin{pmatrix} -a & -\frac{ba}{c} & 0 & \frac{b(a-c)}{c} \\ 0 & a & 0 & c-a \\ 0 & 0 & -c & 0 \\ 0 & 0 & -b & c \end{pmatrix}$$

which lies in I_p. □

Proof of claim 8. Suppose $y \neq \infty, \lambda \neq 0, \infty$. Put $s = 1 - 2\lambda y, t = y, u = 0$ and check that $(D^y_\lambda)^{-1} U_{(s,t,u)} B^{\infty}_{(1,\lambda,\lambda^2)}$ is congruent mod$p$ to

$$\begin{pmatrix} -1 & 0 & \frac{y}{\lambda} & \frac{1-2\lambda y}{2\lambda} \\ \lambda & 1 & -\frac{1}{2\lambda} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2\lambda} \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Now suppose $y \neq \infty$ and put $s = 1, t = -y, u = 0$ and check that

$$(D^y_0)^{-1} U_{(s,t,u)} B^{\infty}_{(1,0,0)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Next put $s = 0, t = -y, u = 1$ and check that $(D^y_\infty)^{-1} U_{(s,t,u)} B^0_{(0,0,1)}$ equals

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now let $\lambda \neq 0, \infty$. Put $s = 1, t = 0, u = 0$ and check that $(D^{\infty}_\lambda)^{-1} U_{(s,t,u)} B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is congruent mod$p$ to

$$\begin{pmatrix} 1 & 0 & -1 & -\frac{1}{2\lambda} \\ 0 & -1 & \frac{1}{2\lambda} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Next put $s = 1, t = 0, u = 0$ and check that $(D^{\infty}_0)^{-1} U_{(s,t,u)} B^0_{(1,0,0)}$ is congruent mod$p$ to

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Finally put $s=0, t=0, u=1$ and check that $(D^{\infty}_{\infty})^{-1}U_{(s,t,u)}B^{\infty}_{(0,0,1)}$ is congruent mod$p$ to

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

□

Proof of claim 9. Suppose two elements f and g in T_m are (l,m)-equivalent. Then there exists $r \in R(\mathbb{Q}_p)$ and $k \in I_p$ such that $rh(l,m)fk = h(l,m)g$. Denote $r' = h(l,m)^{-1}rh(l,m)$ so that $g = r'fk$. Then r' is upper-triangular and belongs to K_p. Writing f, g in block form $f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$, $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$ we conclude as in the proof of Lemma 3.2.1 that the $\mathbb{F}_p$-rank of f_3 equals the $\mathbb{F}_p$-rank of g_3. For class A matrices the $\mathbb{F}_p$-rank of the corresponding 2×2 block is 2 and for class B matrices it is less than 2. So it is not possible that one of f, g is class A and the other class B.

Thus we can assume that f and g are in the same class.

We first deal with the case $m > 0$.

Continuing with the generalities, let $r = tu$ with $T \in \mathbb{Q}_p, u \in U(\mathbb{Q}_p)$. Put

$$t' = h(l,m)^{-1}th(l,m), u' = h(l,m)^{-1}uh(l,m).$$

Thus $r' = t'u'$ and this forces $t' \in (T(\mathbb{Q}_p) \cap K_p), u' \in U(\mathbb{Z}_p)$. We must then have $t = x + zp^m\xi$ with $x \in \mathbb{Z}_p^{\times}, z \in \mathbb{Z}_p$. Let $u' = U_{(s,t,u)}$.

Let us first consider the class A case. We can check that $(A^0_{(0,0,0)})^{-1}t'u'A^{\infty}_{(0,0,0)}$ is congruent mod$p$ to

$$\begin{pmatrix} 0 & x & 0 & 0 \\ x & -zc & 0 & 0 \\ -tx - zcu & -sx - zct & zc & x \\ -ux & -tx & x & 0 \end{pmatrix}$$

and so can never belong to I_p because x is a unit.

We now consider the class B case. Suppose for $(n_1, q_1, r_1), (n_2, q_2, r_2)$ we compute

$$(B^0_{(n_1,q_1,r_1)})^{-1}t'u'B^0_{(n_2,q_2,r_2)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and prove that C is never congruent to 0 (mod p). It will follow then that for any $y_1, y_2 \in V$, $B^{y_1}_{(n_1,q_1,r_1)}$ and $B^{y_2}_{(n_2,q_2,r_2)}$ are not (l,m)-equivalent because the introduction of the new terms Z_{y_1}, Z_{y_2} cannot affect C.

$(B^0_{(1,0,0)})^{-1}t'u'B^0_{(0,0,0)}$ is congruent mod$p$ to

$$\begin{pmatrix} x & zc & sx + zct & tx + zcu \\ 0 & x & tx & ux \\ x & zc & sx + x + zct & tx + zcu \\ 0 & 0 & -zc & x \end{pmatrix}.$$

$(B^0_{(1,0,0)})^{-1}t'u'B^0_{(0,0,1)}$ is congruent mod$p$ to

$$\begin{pmatrix} x & zc-tx-zcu & sx+zct & tx+zcu \\ 0 & x-ux & tx & ux \\ x & zc-tx-zcu & sx+x+zct & tx+zcu \\ 0 & -x & -zc & x \end{pmatrix}.$$

$(B^0_{(0,0,0)})^{-1}t'u'B^0_{(0,0,1)}$ is congruent mod$p$ to

$$\begin{pmatrix} x & zc-tx-zcu & sx+zct & tx+zcu \\ 0 & x-ux & tx & ux \\ 0 & 0 & x & 0 \\ 0 & -x & -zc & x \end{pmatrix}.$$

Each of these three matrices have this property because x is a unit.

Now consider $(B^0_{(1,0,0)})^{-1}t'u'B^\infty_{(1,0,0)}$. This is congruent mod$p$ to

$$\begin{pmatrix} zc & x-sx-zct & tx+zcu & sx+zct \\ x & -tx & ux & tx \\ zc & -sx-zct & tx+zcu & sx+x+zct \\ 0 & zc & x & -zc \end{pmatrix}$$

which cannot belong to I_p because x is a unit.

Next consider $(B^0_{(0,0,0)})^{-1}t'u'B^\infty_{(0,0,0)}$. This is congruent mod$p$ to

$$\begin{pmatrix} zc & x & tx+zcu & sx+zct \\ x & 0 & ux & tx \\ 0 & 0 & 0 & x \\ 0 & 0 & x & -zc \end{pmatrix}$$

which cannot belong to I_p for the same reason.

Finally consider $(B^0_{(0,0,1)})^{-1}t'u'B^\infty_{(0,0,1)}$. This is congruent mod$p$ to

$$\begin{pmatrix} -tx+zc-zcu & x & tx+zcu & sx+zct \\ x-ux & 0 & ux & tx \\ 0 & 0 & 0 & x \\ -ux & 0 & ux+x & tx-zc \end{pmatrix}$$

which can again not belong to I_p.

Thus we have completed the proof of the claim for $m>0$.

For $m=0$ we can only say that $t' = x+z\xi$ with atleast one of x,z an unit.

$(B^0_{(1,0,0)})^{-1}t'u'B^0_{(0,0,0)} =$

$$\begin{pmatrix} x+\frac{1}{2}zb & zc & sx+\frac{1}{2}zb+zct & tx+\frac{1}{2}zb+zcu \\ -za & x-\frac{1}{2}zb & tx-\frac{1}{2}zb-zas & ux-\frac{1}{2}zb-zat \\ x+\frac{1}{2}zb & zc & sx+x-\frac{1}{2}zb+\frac{1}{2}szb+zct & tx+\frac{1}{2}zb+zcu+za \\ 0 & 0 & -zc & x+\frac{1}{2}zb \end{pmatrix}$$

which if in I_p implies $p|z$ which in turn implies $p|x$, a contradiction.

$(B^\infty_{(1,0,0)})^{-1}t'u'B^0_{(0,0,0)} =$

$$\begin{pmatrix} -za & x-\frac{1}{2}zb & -zas+tx-\frac{1}{2}tzb & -zat+ux-\frac{1}{2}uzb \\ x+\frac{1}{2}zb & zc & sx+\frac{1}{2}szb+zct & tx+\frac{1}{2}tzb+zcu \\ 0 & 0 & -zc & x+\frac{1}{2}zb \\ x+\frac{1}{2}zb & zc & sx+x+zct+\frac{1}{2}szb-\frac{1}{2}zb & tx+\frac{1}{2}tzb+zcu+za \end{pmatrix}$$

which cannot belong to I_p for the same reason.

Put $G = z(ct + \frac{1}{2}sb - \frac{1}{2}b)$. $(B^{\infty}_{(1,0,0)})^{-1}t'u'B^{0}_{(1,0,0)} =$

$$\begin{pmatrix} za(s-1) - tx + \frac{1}{2}tzb & x - \frac{1}{2}zb & -zas + tx - \frac{1}{2}tzb & -zat + ux - \frac{1}{2}uzb \\ x(1-s) + G & zc & sx + \frac{1}{2}szb + zct & tx + \frac{1}{2}tzb + zcu \\ zc & 0 & -zc & x + \frac{1}{2}zb \\ \frac{1}{2}zb - G - sx & zc & x(s+1) + G & tx + \frac{1}{2}tzb + zcu + za \end{pmatrix}$$

which cannot belong to I_p for the same reason.

This completes the proof of the final claim. □

3.4. **In which we calculate a certain volume.** For any $t \in K_p$ we define the volume $I_t^{l,m}$ as follows.

$$I_t^{l,m} = \mathrm{vol}(R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(l,m)tI_p). \tag{3.4.1}$$

In this subsection we shall explicitly compute the volume $I_t^{l,m}$. By Proposition 3.3.1, it is enough to do this for $t \in T_m$. The next two propositions state the results and the rest of the section is devoted to proving them.

Proposition 3.4.1. *Let $m > 0$. Let $M_{l,m}$ denote $\frac{p^{3l+4m}}{(p+1)(p^2+1)}$. Then the quantities $I_{t_i}^{l,m}$ for $1 \le i \le 8$ are as follows.*

$$I_{t_1}^{l,m} = pM_{l,m} \qquad I_{t_5}^{l,m} = M_{l,m}$$
$$I_{t_2}^{l,m} = p^2M_{l,m} \qquad I_{t_6}^{l,m} = pM_{l,m}$$
$$I_{t_3}^{l,m} = pM_{l,m} \qquad I_{t_7}^{l,m} = p^2M_{l,m}$$
$$I_{t_4}^{l,m} = M_{l,m} \qquad I_{t_8}^{l,m} = p^3M_{l,m}$$

Proposition 3.4.2. *For $m = 0$ the quantities $I_t^{l,m}$ are as follows.*

$$I_{t_1}^{l,m} = \frac{p^{3l+1}}{(p+1)(p^2+1)} \qquad I_{t_5}^{l,m} = \frac{p^{3l}}{(p+1)(p^2+1)}$$
$$I_{t_2}^{l,m} = \frac{p^{3l+2}}{(p+1)(p^2+1)} \qquad I_{t_7}^{l,m} = \frac{p^{3l+3}}{(p+1)(p^2+1)}$$

Remark. That the volume $I_t^{l,m}$ is finite can be viewed either as a *corollary* of the above propositions, or as a consequence of the fact that $\mathrm{vol}(R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(l,m)K_p)$ is finite [2, section 3].

For each $t \in T_m$ define the subgroup G_t of K_p by

$$G_t = t^{-1}U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)t \cap I_p$$

where $U(\mathbb{Z}_p)$ is the subgroup of K_p consisting of matrices that look like $\begin{pmatrix} 1_2 & M \\ 0 & 1_2 \end{pmatrix}$ with $M = M^T \in M_2(\mathbb{Z}_p)$, and $GL_2(\mathbb{Z}_p)$ (more generally $GL_2(\mathbb{Q}_p)$) is embedded in $G(\mathbb{Q}_p)$ via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & det(g)\cdot(g^{-1})^T \end{pmatrix}$.

Also let $G_t^1 = tG_tt^{-1}$ be the corresponding subgroup of $U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)$.

And finally, define

$$H_t = \{x \in GL_2(\mathbb{Z}_p) \mid \exists y \in U(\mathbb{Z}_p) \text{ such that } yx \in G_t^1\}. \tag{3.4.2}$$

It is easy to see that $H_t = U(\mathbb{Z}_p)G_t^1 \cap GL_2(\mathbb{Z}_p)$, thus H_t is a subgroup of $GL_2(\mathbb{Z}_p)$.

Lemma 3.4.3. *We have a disjoint union*

$$R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(l,m)tI_p = \coprod_{y\in G_t\backslash I_p} R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(l,m)G_t^1 ty.$$

Proof. Since $tI_p = \bigcup_{y\in G_t\backslash I_p} tG_t y = \bigcup_{y\in G_t\backslash I_p} G_t^1 ty$, the only thing to prove is that the union in the statement of the lemma is indeed disjoint.

So suppose that y_1, y_2 are two coset representatives of $G_t\backslash I_p$ and $rh(l,m)g_1ty_1 = h(l,m)g_2ty_2$ with $g_1, g_2 \in G_t^1, r \in R(\mathbb{Q}_p)$.

This means $ty_2y_1^{-1}t^{-1}$ is an element of K_p that is of the form $\begin{pmatrix} A & B \\ 0 & \det(A)\cdot(A^{-1})^T\end{pmatrix}$. Hence $ty_2y_1^{-1}t^{-1} \in U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)$. Thus $y_2y_1^{-1} \in t^{-1}U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)t \cap I_p = G_t$ which completes the proof. □

By the above lemma it follows that

$$I_t^{l,m} = \int_{G_t\backslash I_p} dg \cdot \int_{R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(l,m)G_t^1} dt \tag{3.4.3}$$

$$= p^{3(l+m)}[K_p : I_p]^{-1}[GL_2(\mathbb{Z}_p)U(\mathbb{Z}_p) : G_t^1] \int_{R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(0,m)G_t^1} dt \tag{3.4.4}$$

where we have normalized $\int_{U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)\backslash K_p} dx = 1$.

On the other hand,

$$\begin{aligned} R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(0,m)G_t^1 &= R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(0,m)U(\mathbb{Z}_p)G_t^1 \\ &= R(\mathbb{Q}_p)\backslash R(\mathbb{Q}_p)h(0,m)(U(\mathbb{Z}_p)G_t^1 \cap GL_2(\mathbb{Z}_p)) \\ &= T(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)h(m)H_t \end{aligned}$$

where $h(m) = \begin{pmatrix} p^m & 0 \\ 0 & 1\end{pmatrix}$.

For each $t \in T_m$ let us define

$$A_t = [GL_2(\mathbb{Z}_p)U(\mathbb{Z}_p) : G_t^1]$$

and

$$V_{t,m} = \int_{T(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)h(m)H_t} dt.$$

We use the same normalization of Haar measures as in [2], namely we have

$$\int_{T(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)h(m)GL_2(\mathbb{Z}_p)} dt = 1.$$

We summarize the computations above in the form of a lemma.

Lemma 3.4.4. *Let $m \geq 0$. For each $t \in T_m$ we have*

$$I_t^{l,m} = \frac{p^{3(l+m)}}{(p+1)^2(p^2+1)} \cdot A_t \cdot V_{t,m}.$$

Proof. This follows from equation (3.4.4). □

By exactly the same arguments as in [2, p. 202-203], we see that

$$V_{t,m} = [GL_2(\mathbb{Z}_p) : H_t]^{-1}[T(\mathbb{Z}_p) : O^t_m] \tag{3.4.5}$$

where $O^t_m = T(\mathbb{Q}_p) \cap h(m)H_t h(m)^{-1}$.

Let Γ^0_p (resp. $\Gamma_{0,p}$) be the subgroup of $GL_2(\mathbb{Z}_p)$ consisting of matrices that become lower-triangular (resp. upper-triangular) when reduced mod p.

Lemma 3.4.5. (a) *We have* $H_{t_i} = \Gamma^0_p$ *for* $i = 1, 2, 5, 8$ *and* $H_{t_i} = \Gamma_{0,p}$ *for* $i = 3, 4, 6, 7$.

(b) *The quantities* $A_{t_i} = [U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p) : G^1_{t_i}]$ *are as follows:*

$$\begin{aligned} A_{t_1} &= p(p+1) & A_{t_5} &= p+1 \\ A_{t_2} &= p^2(p+1) & A_{t_6} &= p+1 \\ A_{t_3} &= p^2(p+1) & A_{t_7} &= p^3(p+1) \\ A_{t_4} &= p(p+1) & A_{t_8} &= p^3(p+1) \end{aligned}$$

Proof. We will prove this directly using (3.4.2) and the definition of A_{t_i}.

First observe that the cardinality of $U(\mathbb{F}_p)GL_2(\mathbb{F}_p)$ is $p^3 \cdot (p^2 - p)(p^2 - 1) = p^4(p-1)^2(p+1)$. Recall also that the images of Γ^0_p and $\Gamma_{0,p}$ have cardinality $p(p-1)^2$ in $GL_2(\mathbb{F}_p)$.

Suppose

$$U = \begin{pmatrix} 1 & 0 & n & q \\ 0 & 1 & q & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, G = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{pmatrix}.$$

We have

$$t_1^{-1}UGt_1 = \begin{pmatrix} a - nd + qb & b & nd - qb & -nc + qa \\ c - qd + rb & d & qd - rb & -qc + ra \\ a - nd + qb - d & b & nd - qb + d & -nc + qa - c \\ b & 0 & -b & a \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $b \equiv 0 \pmod p, n \equiv \frac{a}{d} - 1 \pmod p$. So $H_{t_1} = \Gamma^0_p$ and $A_{t_1} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p^2} = p(p+1)$.

$$t_2^{-1}UGt_2 = \begin{pmatrix} d & c - qd + rb & ra - qc & qd - rb \\ b & a - nd + qb & qa - nc & nd - qb \\ 0 & b & a & -b \\ b & a - nd + qb - d & qa - c - nc & nd - qb + d \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $b \equiv 0 \pmod p, n \equiv \frac{a}{d} - 1 \pmod p, q \equiv \frac{c}{d}$. So $H_{t_2} = \Gamma^0_p$ and $A_{t_2} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p} = p^2(p+1)$.

$$t_3^{-1}UGt_3 = \begin{pmatrix} a & b + nc - qa & nd - qb & -nc + qa \\ c & d + qc - ra & qd - rb & ra - qc \\ 0 & c & d & -c \\ c & d + qc - ra - a & qd - rb - b & ra - qc + a \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $c \equiv 0 \pmod p, r \equiv \frac{d}{a} - 1 \pmod p, q \equiv \frac{b}{a}$. So $H_{t_3} = \Gamma_{0,p}$ and $A_{t_3} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p} = p^2(p+1)$.

$$t_4^{-1}UGt_4 = \begin{pmatrix} d+qc-ra & c & ra-qc & qd-rb \\ b+nc-qa & a & qa-nc & nd-qb \\ d+qc-ra-a & c & ra-qc+a & qd-rb-b \\ c & 0 & -c & d \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $c \equiv 0 \pmod p, r \equiv \frac{d}{a} - 1 \pmod p$. So $H_{t_4} = \Gamma_{0,p}$ and $A_{t_4} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p^2} = p(p+1)$.

$$t_5^{-1}UGt_5 = \begin{pmatrix} a & b & nd-qb & -nc+qa \\ c & d & qd-rb & -qc+ra \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $b \equiv 0 \pmod p$. So $H_{t_5} = \Gamma_p^0$ and $A_{t_5} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p^3} = (p+1)$.

$$t_6^{-1}UGt_6 = \begin{pmatrix} d & c & ra-qc & qd-rb \\ b & a & qa-nc & nd-qb \\ 0 & 0 & a & -b \\ 0 & 0 & -c & d \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $c \equiv 0 \pmod p$. So $H_{t_6} = \Gamma_{0,p}$ and $A_{t_6} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p^3} = (p+1)$.

$$t_7^{-1}UGt_7 = \begin{pmatrix} d & -c & 0 & 0 \\ -b & a & 0 & 0 \\ qb-nd & nc-qa & a & b \\ rb-qd & qc-ra & c & d \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $c \equiv 0 \pmod p, n \equiv 0 \pmod p, q \equiv 0 \pmod p, r \equiv 0 \pmod p$. So $H_{t_7} = \Gamma_{0,p}$ and $A_{t_7} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2} = p^3(p+1)$.

$$t_8^{-1}UGt_8 = \begin{pmatrix} a & -b & 0 & 0 \\ -c & d & 0 & 0 \\ qc-ra & rb-qd & d & c \\ nc-qa & qb-nd & b & a \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $b \equiv 0 \pmod p, n \equiv 0 \pmod p, q \equiv 0 \pmod p, r \equiv 0 \pmod p$. So $H_{t_8} = \Gamma_p^0$ and $A_{t_8} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2} = p^3(p+1)$.

□

Let t be such that $H_t = \Gamma_p^0$. Then by working through the definitions, we see that

$$O_m^t = x + p^{m+1}y\xi_0, \; x, y \in \mathbb{Z}_p. \tag{3.4.6}$$

On the other hand if t is such that $H_t = \Gamma_{0,p}$, then we see that

$$O_m^t = x + p^{m}y\xi_0, \; x, y \in \mathbb{Z}_p. \tag{3.4.7}$$

Lemma 3.4.6. *Let $m > 0$. Then we have $V_{t_i,m} = p^m$ for $i = 1, 2, 5, 8$ and $V_{t_i,m} = p^{m-1}$ for $i = 3, 4, 6, 7$.*

Proof. This follows from (3.4.5), (3.4.6), (3.4.7), Lemma 3.4.5 and [2, Lemma 3.5.3] □

Proof of Proposition 3.4.1. The proof is a consequence of Lemma 3.4.4, Lemma 3.4.5 and Lemma 3.4.6. □

Let us now look at the case $m = 0$. In this case $T_0 = \{t_1, t_2, t_5, t_7\}$.

The groups H_{t_i} and the quantities $[GL_2(\mathbb{Z}_p) : H_{t_i}]^{-1}$ have already been calculated. On the other hand we now have

$$O_0^{t_i} = x + py\xi_0,\ x, y \in \mathbb{Z}_p. \tag{3.4.8}$$

for each $t_i \in T_0$.

Proof of Proposition 3.4.2. We have already calculated each A_{t_i}. Also by (3.4.5),(3.4.8) and Lemma 3.4.5 we conclude that each $V_{t_i,0} = 1$. Now the result follows as before, from Lemma 3.4.4.

□

3.5. **Simplification of the local zeta integral.** Recall the definition of the key local integral $Z_p(s)$ from section 2. In (3.3.3) we reduced this integral to an useful sum. Now suppose that W_p and B_p are right I_p-invariant. Then proposition 3.3.1 allows us to further simplify that expression as follows.

$$Z_p(s) = \sum_{l \in \mathbb{Z}, m \geq 0} \sum_{t \in T_m} W_p(\Theta h(l,m)t, s) \cdot B_p(h(l,m)t) \cdot I_t^{l,m} \tag{3.5.1}$$

Note that in the above formula we mildly abuse notation and use Θ to really mean its natural inclusion in $\widetilde{G}(\mathbb{Q}_p)$. We will continue to do this in the future for notational economy.

Remark. The importance of section 3.4, where we calculated $I_t^{l,m}$ for each $t \in T_m$, is that we can now use the formula (3.5.1) to evaluate the local zeta integral whenever the local functions W_p and B_p can be explicitly determined.

4. The evaluation of the local Bessel functions in the Steinberg case

4.1. **Background.** Because automorphic representations of $GSp(4)$ are not necessarily generic, the Whittaker model is not always useful for studying L-functions. For many problems, the Bessel model is a good substitute. Explicit evaluation of local zeta integrals then often reduces to explicit evaluation of certain local Bessel functions. Formulas for the Bessel functions have been established in the following cases.

[23] *unramified* representations of $GSp_4(\mathbb{Q}_p)$
[1] *unramified* representations (the Casselman-Shalika like formula)
[16] *class-one* representations on $Sp_4(\mathbb{R})$
[15] large *discrete series* and P_J*-principal series* of $Sp_4(\mathbb{R})$
[12] *principal series* of $Sp_4(\mathbb{R})$

In this section we give an explicit formula for the Bessel function for an unramified quadratic twist of the *Steinberg* representation of $GSp_4(\mathbb{Q}_p)$. By [21] this is precisely the representation corresponding to a local newform for the Iwahori subgroup.

Throughout this section we let p be an odd prime that is inert in L. We suppose that the local component $(\omega_\pi)_p$ is trivial, the conductor of ψ_p is $\mathbb{Z}_p$ and $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_p)$.

Because p is inert, L_p is a quadratic extension of $\mathbb{Q}_p$ and we may write elements of L_p in the form $a + b\sqrt{-d}$ with $a, b \in \mathbb{Q}_p$; then $\mathbb{Z}_{L,p} = a + b\sqrt{-d}$ where $a, b \in \mathbb{Z}_p$. We identify L_p with $T(\mathbb{Q}_p)$ and ξ with $\sqrt{-d}/2$. Then $T(\mathbb{Z}_p) = \mathbb{Z}_{L,p}^\times$ consists of elements of the form $a + b\sqrt{-d}$ where a, b are elements of $\mathbb{Z}_p$ not both divisible by p.

We assume that Λ_p is trivial on the elements of $T(\mathbb{Z}_p)$ of the form $a + b\sqrt{-d}$ with $a, b \in \mathbb{Z}_p, p \mid b, p \nmid a$. Further, we assume that Λ_p is *not* trivial on the full group $T(\mathbb{Z}_p)$, that is, it is not unramified.

Finally, assume that the local representation π_p is an unramified twist of the Steinberg representation. This is representation IVa in [21, Table 1]. The space of π_p contains a unique normalized vector that is fixed by the Iwahori subgroup I_p. We can think of this vector as the normalized local newform for this representation.

4.2. **Bessel functions.** Let $\mathfrak{B}$ be the space of locally constant functions φ on $G(\mathbb{Q}_p)$ satisfying

$$\varphi(tuh) = \Lambda_p(t)\theta_p(u)\varphi(h), \text{ for } t \in T(\mathbb{Q}_p), u \in U(\mathbb{Q}_p), h \in G(\mathbb{Q}_p).$$

Then by Novodvorsky and Piatetski-Shapiro [17], there exists a unique subspace $\mathfrak{B}(\pi_p)$ of $\mathfrak{B}$ such that the right regular representation of $G(\mathbb{Q}_p)$ on $\mathfrak{B}(\pi_p)$ is isomorphic to π_p. Let B_p be the unique I_p-fixed vector in $\mathfrak{B}(\pi_p)$ such that $B_p(1_4) = 1$. Therefore

$$B_p(tuhk) = \Lambda_p(t)\theta_p(u)\varphi(h), \tag{4.2.1}$$

where $t \in T(\mathbb{Q}_p), u \in U(\mathbb{Q}_p), h \in G(\mathbb{Q}_p), k \in K_p$.

Our goal is to explicitly compute B_p. By Proposition 3.3.1 and (4.2.1) it is enough to compute the values $B_p(h(l,m)t_i)$ for $l \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}, t_i \in T_m$.

Let us fix some notation. Recall the matrices t_i which were defined in Subsection 3.3. Also we will frequently use other notation from Section 3. We now define

$$\begin{aligned} a_0^{l,m} &= B_p(h(l,m)t_7), & a_\infty^{l,m} &= B_p(h(l,m)t_8), \\ b_0^{l,m} &= B_p(h(l,m)t_2), & {}^1b_0^{l,m} &= B_p(h(l,m)t_1), \\ b_\infty^{l,m} &= B_p(h(l,m)t_3), & {}^1b_\infty^{l,m} &= B_p(h(l,m)t_4), \\ c_0^{l,m} &= B_p(h(l,m)t_5), & c_\infty^{l,m} &= B_p(h(l,m)t_6). \end{aligned}$$

Lemma 4.2.1. *Let $m \geq 0, y \in \{0, \infty\}$. The following equations hold:*

(a) $a_y^{l,m} = 0 \quad$ *if* $l < -1$.
(b) ${}^1b_0^{l,m} = b_0^{l,m} = {}^1b_\infty^{l,0} = b_\infty^{l,0} = 0 \quad$ *if* $l < 0$.
(c) ${}^1b_\infty^{l,m} = b_\infty^{l,m} = 0 \quad$ *if* $l < -1$.
(d) $c_y^{l,m} = 0 \quad$ *if* $l < 0$.

Proof. First note that $U_{(0,0,p)}t_i \equiv t_i \pmod p$, hence they are in the same coset of K_p/I_p. Hence

$$\begin{aligned} B_p(h(l,m)t_i) &= B_p(h(l,m)U_{(0,0,p)}t_i)\\ &= B_p(U_{(0,0,p^{l+1})}h(l,m)t_i)\\ &= \psi_p(p^{l+1}c)B_p(h(l,m)t_i). \end{aligned}$$

Since the conductor of ψ_p is $\mathbb{Z}_p$ and c is a unit, it follows that $B_p(h(l,m)t_i)=0$ for $l<-1$. This completes the proof of (a) and (c).

Next, observe that

$$\begin{aligned} c_y^{l,m} &= B_p(h(l,m)Z_y)\\ &= B_p(h(l,m)U_{(0,0,1)}Z_y)\\ &= B_p(U_{(0,0,p^l)}h(l,m)Z_y)\\ &= \psi_p(p^lc)B_p(h(l,m)Z_y). \end{aligned}$$

It follows that $B_p(h(l,m)Z_y)=0$ for $l<0$. This completes the proof of (d).

Next, we have

$$\begin{aligned} B_p(h(l,m)JU_{(1,0,0)}JZ_y) &= B_p(h(l,m)JU_{(1,0,0)}JU_{0,0,1}Z_y)\\ &= B_p(h(l,m)U_{0,0,1}JU_{(1,0,0)}JZ_y)\\ &= \psi_p(p^lc)B_p(h(l,m)JU_{(1,0,0)}JZ_y). \end{aligned}$$

It follows that ${}^1b_0^{l,m} = b_0^{l,m} = 0$ for $l<0$.

Finally,

$$\begin{aligned} B_p(h(l,0)JU_{(0,0,1)}JZ_y) &= B_p(h(l,0)JU_{(0,0,1)}JU_{1,0,0}Z_y)\\ &= B_p(h(l,0)U_{1,0,0}JU_{(0,0,1)}JZ_y)\\ &= \psi_p(p^la)B_p(h(l,0)JU_{(0,0,1)}JZ_y). \end{aligned}$$

It follows that ${}^1b_\infty^{l,0} = b_\infty^{l,0} = 0$ for $l<0$. This completes the proof of (b). □

By our normalization, we have $c_0^{0,0}=1$. From Proposition 3.3.1, proof of Claim 6, it follows that $c_\infty^{0,0} = \Lambda_p(\frac{b+\sqrt{-d}}{2})$.

To get more information, we have to use the fact that the local Iwahori-Hecke algebra acts on B_p in a precise manner.

4.3. **Hecke operators and the results.** Henceforth we always assume that $l\geq -1, m\geq 0$. In particular, all equations that are stated without qualification will be understood to hold in the above range. We know that π_p is either $\mathrm{St}_{GSp(4)}$ or $\xi_0\mathrm{St}_{GSp(4)}$ where ξ_0 is the non-trivial unramified quadratic character. Put $w_p=-1$ in the former case and $w_p=1$ in the latter. Put

$$\eta_p = \begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&p&0&0\\ p&0&0&0 \end{pmatrix}.$$

Also, for $y\in V$, define the matrices R_y as follows: If $y\in Y$,

$$R_y = (U_{(y,0,0)})^T,$$

and

$$R_\infty = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $t \in G(\mathbb{Q}_p)$. By [21], we know the following:

$$\sum_{y \in V} B_p(tZ_y) = 0, \tag{4.3.1}$$

$$B_p(t\eta_p) = w_p B_p(t), \tag{4.3.2}$$

$$\sum_{y \in V} B_p(tR_y) = 0. \tag{4.3.3}$$

(4.3.1) and Proposition 3.3.1 immediately imply

$$a_0^{l,m} + pa_\infty^{l,m} = 0, \qquad \text{for } m > 0 \tag{4.3.4}$$

$$pb_y^{l,m} + {}^1b_y^{l,m} = 0, \qquad \text{for } y \in \{0, \infty\} \tag{4.3.5}$$

$$pc_0^{l,m} + c_\infty^{l,m} = 0, \qquad \text{for } m > 0 \tag{4.3.6}$$

Next we act upon by η_p. Check that

$$(h(l+1,m)B^\infty_{(0,0,0)})^{-1} h(l,m) A^0_{(0,0,0)} \eta_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

So we have

$$\begin{aligned} a_0^{l,m} &= B_p(h(l,m)A^0_{(0,0,0)}) \\ &= w_p B_p(h(l,m)A^0_{(0,0,0)}\eta_p) \\ &= w_p B_p(h(l+1,m)B^\infty_{(0,0,0)}). \end{aligned}$$

Thus

$$a_0^{l,m} = w_p c_\infty^{l+1,m}. \tag{4.3.7}$$

We also have

$$(h(l+1,m)B^0_{(0,0,0)})^{-1} h(l,m) A^\infty_{(0,0,0)} \eta_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

So similarly, we conclude

$$a_\infty^{l,m} = w_p c_0^{l+1,m}. \tag{4.3.8}$$

Next, check that

$$(h(l,m)B^1_{(1,0,0)}\eta_p)^{-1} h(l-1,m+1) U_{(-1/p,0,0)} D^1_\infty = (Z^1)^T \in I_p.$$

Hence

$$B_p(h(l,m)B^1_{(1,0,0)}) = w_p B_p(h(l-1,m+1)D^1_\infty).$$

(Note that both sides are zero if $l = -1, m = 0$).

By the proof of Proposition 3.3.1, $B_p(h(l,m)B^1_{(1,0,0)}) = b_0^{l,m}$ and $B_p(h(l-1,m+1)D^1_\infty) = \psi_p(p^{l-1}c)b_\infty^{l-1,m+1}$.

Thus we have proved

$$b_0^{l,m} = w_p\psi_p(p^{l-1}c)b_\infty^{l-1,m+1}. \tag{4.3.9}$$

At this point we pause and note that on account of (4.3.4)–(4.3.9) it is enough to compute the quantities $b_\infty^{l,m}, a_0^{l,m}, l \geq -1, m \geq 0, l+m \neq -1$. Of course, we already know that $a_0^{-1,0} = w_p\Lambda_p(\frac{b+\sqrt{-d}}{2})$.

Next, we use (4.3.3).

For each $x \in Y$, we can check that $A^0_{0,0,0}R_x = A^0_{-x,0,0}$. Furthermore, $A^0_{0,0,0}R_\infty = D^0_\infty$. Assuming $l+m \geq 0$ we have $B_p(h(l,m)A^0_{-x,0,0}) = a_0^{l,m}$ and $B_p(h(l,m)D^0_\infty = \psi_p(p^lc)b_\infty^{l,m}$. So using (4.3.3) we conclude

$$pa_0^{l,m} = -\psi_p(p^lc)b_\infty^{l,m}, \tag{4.3.10}$$

for $l+m \geq 0$.

However we can do more. Check that for $x \in Y$, $A^\infty_{(0,0,0)}R_x = A^\infty_{(0,0,-x)}$ and $A^\infty_{(0,0,0)}R_\infty \equiv D^0_0 \pmod p$. If $l \geq 0$ we have $B_p(h(l,m)A^\infty_{(0,0,-x)} = a_\infty^{l,m}$ and $B_p(h(l,m)D^0_0) = b_0^{l,m}$. So again using (4.3.3) we have

$$pa_\infty^{l,m} = -b_0^{l,m}, \tag{4.3.11}$$

for $l \geq 0$.

So (4.3.4), (4.3.9) and (4.3.11) imply that for $l \geq 0$, $m > 0$

$$b_\infty^{l,m} = -pb_0^{l,m} = -pw_p\psi_p(p^{l-1}c)b_\infty^{l-1,m+1}. \tag{4.3.12}$$

Now observe that $B^0_{0,0,0}R_\infty \equiv D^\infty_0 \pmod p$ and for $x \in Y$, $B^0_{0,0,0}R_x = B^0_{-x,0,0}$. Assuming $l+m \neq -1$ we have $B_p(h(l,m)D^\infty_0) = {}^1b_0^{l,m}$ and for $x \in y, x \neq 0$, $B_p(h(l,m)B^0_{-x,0,0}) = {}^1b_0^{l,m}$. Hence using (4.3.3)

$$c_0^{l,m} = -p\ {}^1b_0^{l,m} \tag{4.3.13}$$

So by equations (4.3.5) and (4.3.8) we have,

$$a_\infty^{l,m} = p^2\psi_p(p^lc)b_\infty^{l,m+1} \tag{4.3.14}$$

The above equation, along with our normalization tells us that

$$b_\infty^{-1,1} = \frac{1}{p^2}\psi_p(-\frac{c}{p})w_p. \tag{4.3.15}$$

Also, using (4.3.11),(4.3.12) and (4.3.14) we get

$$b_\infty^{l,m+1} = \frac{1}{p^4}b_\infty^{l,m} \tag{4.3.16}$$

for $l \geq 0, m > 0$.

(4.3.12),(4.3.16) and(4.3.15) imply :

$$b_\infty^{l,m} = -\frac{(-pw_p)^l}{p^{4l+4m+1}} \quad \text{if } l \geq 0, m \geq 1 \tag{4.3.17}$$

$$b_\infty^{-1,m} = \frac{1}{p^{4m-2}}\psi_p(-\frac{c}{p})w_p \quad \text{if } m \geq 1. \tag{4.3.18}$$

In the case $m = 0$, Proposition 3.3.1, proof of Claim 7, tells us that ${}^1b_\infty^{l,0} = \Lambda_p(\frac{b+\sqrt{-d}}{2})\ {}^1b_0^{l,0}$ which implies

$$b_\infty^{l,0} = \Lambda_p(\frac{b+\sqrt{-d}}{2})b_0^{l,0} = w_p\psi_p(p^{l-1}c)\Lambda_p(\frac{b+\sqrt{-d}}{2})b_\infty^{l-1,1} \tag{4.3.19}$$

Equation (4.3.17)–(4.3.19), along with the earlier equations that specify the inderdependence of various quantities, determine all the values $B_p(h(l,m)t_i)$. For convenience, we compactly state the facts proven above above as two propositions. We only state it for $l \geq 0$ since that is the only case needed for our later applications. The values for $l = -1$ can be easily gleaned from these and the above equations.

Proposition 4.3.1. *Let $l \geq 0, m > 0$. Put $M = (-pw_p)^l p^{-4(l+m)}$. Then the following hold:*

(a) $B_p(h(l,m)t_1) = M \cdot \frac{-1}{p}$,
(b) $B_p(h(l,m)t_2) = M \cdot \frac{1}{p^2}$,
(c) $B_p(h(l,m)t_3) = M \cdot \frac{-1}{p}$,
(d) $B_p(h(l,m)t_4) = M$
(e) $B_p(h(l,m)t_5) = M$
(f) $B_p(h(l,m)t_6) = M \cdot (-p)$,
(g) $B_p(h(l,m)t_7) = M \cdot \frac{1}{p^2}$,
(h) $B_p(h(l,m)t_8) = M \cdot \frac{-1}{p^3}$.

Proposition 4.3.2. *Let $l \geq 0$. Put $M = (-pw_p)^l p^{-4l}$. Then the following hold:*

(a) $B_p(h(l,0)t_1) = M \cdot \frac{-1}{p}$,
(b) $B_p(h(l,0)t_2) = M \cdot \frac{1}{p^2}$,
(c) $B_p(h(l,0)t_5) = M$,
(d) $B_p(h(l,m)t_7) = M \cdot \frac{-\Lambda_p(\frac{b+\sqrt{-d}}{2})}{p^3}$.

5. The case unramified π_p, Steinberg σ_p

5.1. **Assumptions.** Suppose that the characters $\omega_\pi, \omega_\sigma, \chi_0$ are trivial. Let $p \neq 2$ be a finite prime of $\mathbb{Q}$ such that

(a) p is inert in $L = \mathbb{Q}(\sqrt{-d})$.
(b) The local components Λ_p and π_p are unramified.
(c) σ_p is the Steinberg representation (or its twist by the unramified quadratic character).
(d) The conductor of ψ_p is $\mathbb{Z}_p$
(e) $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_p)$.
(f) $-d = b^2 - 4ac$ generates the discriminant of $L_p/\mathbb{Q}_p$.

Remark: σ_p is concretely realized as (possibly the unramified quadratic twist of) the special representation on the locally constant functions of $B_p\backslash GL_2(\mathbb{Q}_p)$ modulo the constant functions (where B_p is the standard Borel subgroup consisting of upper-triangular matrices). It corresponds to the local newform for the Iwahori subgroup $\Gamma_0(p)$ of $GL_2(\mathbb{Q}_p)$.

5.2. **Description of B_p and W_p.** Given the local representations and characters as above, define $I(\Pi_p, s)$ and the local Bessel and Whittaker spaces as in Sections 1 and 2. For any choice of local Whittaker and Bessel functions W_p and B_p we can define the local zeta integral $Z_p(s)$ by (2.2.3). We now fix such a choice.

As in the unramified case from section 1, we let B_p be the unique normalized K_p-vector in the local Bessel space. Sugano[23] has computed the function B_p explicitly.

We now define W_p. Let $\widetilde{U_p}$ be the subgroup of $\widetilde{K_p}$ defined by

$$\widetilde{U_p} = \left\{ z \in \widetilde{K_p} \mid z \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \pmod{p} \right\}.$$

It is not hard to see that $I(\Pi_p, s)$ has $\widetilde{U_p}$-fixed vectors. Now let W_p be the unique $\widetilde{U_p}$-fixed vector in the local Whittaker space with the following properties:

- $W_p(e, s) = 1$,
- $W_p(g, s) = 0$ if g does not belong to $P(\mathbb{Q}_p)\widetilde{U_p}$

Concretely we have the following description of $W_p(s)$.

We know that $\sigma_p = \mathrm{Sp} \otimes \tau$ where Sp denotes the special (Steinberg) representation and τ is a (possibly trivial) unramified quadratic character. We put $a_p = \tau(p)$, thus $a_p = \pm 1$ is the eigenvalue of the local Hecke operator $T(p)$.

Let W'_p be the unique function on $GL_2(\mathbb{Q}_p)$ such that

$$W'_p(gk) = W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), k \in \Gamma_{0,p}, \tag{5.2.1}$$

$$W'_p(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \psi_p(-cx)W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), x \in \mathbb{Q}_p, \tag{5.2.2}$$

$$W'_p \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} \tau(a)|a| & \text{if } |a|_p \le 1, \\ 0 & \text{otherwise} \end{cases} \tag{5.2.3}$$

$$W'_p \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \begin{cases} -p^{-1}\tau(a)|a| & \text{if } |a|_p \le p, \\ 0 & \text{otherwise} \end{cases} \tag{5.2.4}$$

We extend W'_p to a function on $GU(1,1)(\mathbb{Q}_p)$ by

$$W'_p(ag) = W'_p(g), \text{ for } a \in L_p^\times, g \in GL_2(\mathbb{Q}_p).$$

Then, $W_p(s)$ is the unique function on $\widetilde{G}(\mathbb{Q}_p)$ such that

$$W_p(mnk, s) = W_p(m, s), \text{ for } m \in M(\mathbb{Q}_p), n \in N(\mathbb{Q}_P), k \in \widetilde{U_p}, \tag{5.2.5}$$

$$W_p(e) = 1 \text{ and } W_p(g, s) = 0 \text{ if } g \notin P(\mathbb{Q}_p)\widetilde{U_p}, \tag{5.2.6}$$

and

$$W_p\left(\begin{pmatrix} a & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \overline{a}^{-1} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a_1 & 0 & b_1\\ 0 & 0 & c_1 & 0\\ 0 & d_1 & 0 & e_1\end{pmatrix}, s\right) \tag{5.2.7}$$
$$= \left|N_{L/\mathbb{Q}}(a)\cdot c_1^{-1}\right|_p^{3(s+1/2)}\cdot W_p'\begin{pmatrix} a_1 & b_1\\ d_1 & e_1\end{pmatrix}$$

for $a \in \mathbb{Q}_p^\times$, $\begin{pmatrix} a_1 & b_1\\ d_1 & e_1\end{pmatrix} \in GU(1,1)(\mathbb{Q}_p)$, $c_1 = \mu_1\begin{pmatrix} a_1 & b_1\\ d_1 & e_1\end{pmatrix}$.

Let us use the following notation: For $\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in GU(1,1)$ we let

$$m^{(2)}(\begin{pmatrix} a & b\\ c & d\end{pmatrix}) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a & 0 & b\\ 0 & 0 & \beta & 0\\ 0 & c & 0 & d\end{pmatrix}$$

where $\beta = \mu_1(\begin{pmatrix} a & b\\ c & d\end{pmatrix})$.

5.3. **The results.** For $i = 1,2,3,4$, define the characters $\gamma_p^{(i)}$ of $\mathbb{Q}_p^\times$ as in Section 1. We now state and prove the main theorem of this section.

Theorem 5.3.1. *Let the functions B_p, W_p be as defined in subsection 5.2. Then we have*

$$Z_p(s, W_p, B_p) = \frac{1}{p^2+1}\cdot\frac{L(3s+\frac{1}{2}, \pi_p\times\sigma_p)}{L(3s+1, \sigma_p\times\rho(\Lambda_p))}$$

where,

$$L(s, \pi_p\times\sigma_p) = \prod_{i=1}^{4}(1-\gamma_p^{(i)}(p)a_p p^{-1/2}p^{-s})^{-1},$$

and

$$L(s, \sigma_p\times\rho(\Lambda_p)) = (1-p^{-2s-1})^{-1}.$$

Before we begin the proof, we need a lemma.

Lemma 5.3.2. *We have the following formulae for $W_p(\Theta h(l,m)t_i, s)$ where $t_i \in T_m$.*

(a) *If $m > 0$ then* $W_p(\Theta h(l,m)t_i, s) = \begin{cases} p^{-6ms-3ls-3m-5l/2}a_p^l & \text{if } i \in \{1,5\}\\ p^{-6ms-3ls-3m-5l/2}a_p^l\cdot\frac{-1}{p} & \text{if } i\in\{3,7\}\\ 0 & \text{otherwise}\end{cases}$

(b) $W_p(\Theta h(l,0)t_i, s) = \begin{cases} p^{-3ls-5l/2}a_p^l & \text{if } i\in\{1,5\}\\ 0 & \text{if } i \in \{2,7\}\end{cases}$

Proof. We have

$$\Theta h(l,m) = h(l,m)\begin{pmatrix} 1 & 0 & 0 & 0\\ p^m\alpha & 1 & 0 & 0\\ 0 & 0 & 1 & -p^m\overline{\alpha}\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{5.3.1}$$

First consider the case $m > 0$.

We claim that $\Theta h(l,m)t_i \notin P(\mathbb{Q}_p)\widetilde{U_p}$ if $i \in \{2,4,6,8\}$.

Put $\Theta_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p^m\alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & -p^m\overline{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Using (6.3.1), it suffices to prove that $\Theta_m t_i \notin P(\mathbb{Z}_p)\widetilde{U_p}$. So take a typical element element

$$P = \begin{pmatrix} a & ax & at + ax\overline{y} & ay \\ 0 & m & m\overline{y} - \beta\overline{x} & \beta \\ 0 & 0 & \lambda/\overline{a} & 0 \\ 0 & \gamma & \gamma\overline{y} - \delta\overline{x} & \delta \end{pmatrix} \in P(\mathbb{Z}_p) \tag{5.3.2}$$

where all the variables lie in $\mathbb{Z}_L$ and $\begin{pmatrix} m & \beta \\ \gamma & \delta \end{pmatrix} \in GU(1,1)(\mathbb{Z}_p)$ with $\lambda = \mu_1 \begin{pmatrix} m & \beta \\ \gamma & \delta \end{pmatrix}$.

We have

$$P\Theta_m t_2 = \begin{pmatrix} ax & a + axp^m\alpha - at - ax\overline{y} & -(at+ax\overline{y})\overline{\alpha}p^m + ay & at + ax\overline{y} \\ m & mp^m\alpha - m\overline{y} + \beta\overline{x} & -(m\overline{y} - \beta\overline{x})\overline{(\alpha)}p^m + \beta & m\overline{y} - \beta\overline{x} \\ 0 & -\lambda(\overline{a})^{-1} & -\overline{\alpha}p^m\lambda(\overline{a})^{-1} & \lambda(\overline{a})^{-1} \\ \gamma & \gamma p^m\alpha - \gamma\overline{y} + \delta\overline{x} & -(\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m + \delta & \gamma\overline{y} - \delta\overline{x} \end{pmatrix}$$

which, if it were an element of $\widetilde{U_p}$ would imply that $p \mid \lambda$, a contradiction.

We have

$$P\Theta_m t_4 = \begin{pmatrix} ax + (at+ax\overline{y})\overline{\alpha}p^m - ay & a + axp^m\alpha & -(at+ax\overline{y})\overline{\alpha}p^m + ay & at + ax\overline{y} \\ m + (m\overline{y} - \beta\overline{x})\overline{\alpha}p^m - \beta & mp^m\alpha & -(m\overline{y} - \beta\overline{x})\overline{\alpha}p^m + \beta & m\overline{y} - \beta\overline{x} \\ \overline{\alpha}p^m\lambda(\overline{a})^{-1} & 0 & -\overline{\alpha}p^m\lambda(\overline{a})^{-1} & \lambda(\overline{a})^{-1} \\ \gamma + (\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m - \delta & \gamma p^m\alpha & -(\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m + \delta & \gamma\overline{y} - \delta\overline{x} \end{pmatrix}$$

which, if it were an element of $\widetilde{U_p}$ would imply that $p \mid \lambda$, a contradiction.

We have

$$P\Theta_m t_6 = \begin{pmatrix} ax & a + axp^m\alpha & -(at+ax\overline{y})\overline{\alpha}p^m + ay & at + ax\overline{y} \\ m & mp^m\alpha & -(m\overline{y} - \beta\overline{x})\overline{\alpha}p^m + \beta & m\overline{y} - \beta\overline{x} \\ 0 & 0 & -\overline{\alpha}p^m\lambda(\overline{a})^{-1} & \lambda(\overline{a})^{-1} \\ \gamma & \gamma p^m\alpha & -(\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m + \delta & \gamma\overline{y} - \delta\overline{x} \end{pmatrix}$$

which, if it were an element of $\widetilde{U_p}$ would imply that $p \mid \gamma, p \mid \delta$, a contradiction.

Finally we have

$$P\Theta_m t_8 = \begin{pmatrix} (at+ax\overline{y})\overline{\alpha}p^m - ay & -at - ax\overline{y} & ax & a + axp^m\alpha \\ (m\overline{y} - \beta\overline{x})\overline{\alpha}p^m - \beta & -m\overline{y} + \beta\overline{x} & m & mp^m\alpha \\ \lambda\overline{\alpha}p^m(\overline{a})^{-1} & -\lambda(\overline{a})^{-1} & 0 & 0 \\ (\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m - \delta & -\gamma\overline{y} + \delta\overline{x} & \gamma & \gamma p^m\alpha \end{pmatrix}$$

which, if it were an element of $\widetilde{U_p}$ would imply that $p \mid \gamma, p \mid \delta$, a contradiction. This completes the proof of the claim.

For the remaining t_i (i.e. $i \in \{1,3,5,7\}$) we have the following decompositions:

$\Theta h(l,m)t_1$

$$= \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ 0 & p^m \end{pmatrix}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ -p^m\alpha & -1 & 0 & 0 \\ 1 & 0 & -1 & p^m\overline{\alpha} \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\Theta h(l,m)t_3 =$

$$\begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ -p^m & p^m \end{pmatrix}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ -p^m\alpha & -1 & 0 & 0 \\ 0 & -p^m\overline{\alpha} & -1 & p^m\overline{\alpha} \\ -p^m\alpha & 0 & 0 & -1 \end{pmatrix}$$

$\Theta h(l,m)t_5$

$$= \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ 0 & p^m \end{pmatrix}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ p^m\alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & p^m\overline{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Theta h(l,m)t_7$

$$= \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} 0 & -p^{m+l} \\ -p^m & 0 \end{pmatrix}) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & p^m\overline{\alpha} & 0 & 0 \\ 0 & 0 & p^m\alpha & 1 \end{pmatrix}$$

Part (a) of the lemma now follows from the above decompositions and equations(5.2.1)-(5.2.7).

Let us now look at $m = 0$. Once again, let P be the matrix defined in (5.3.2)The same proof as above for t_2 shows that $P\Theta_m t_2 \notin \widetilde{U_p}$. As for t_7,

$$P\Theta_m t_7 = \begin{pmatrix} -at - ax\overline{y} & (at + ax\overline{y})\overline{\alpha} - ay & a + ax\alpha & ax \\ -m\overline{y} + \beta\overline{x} & (m\overline{y} - \beta\overline{x})\overline{\alpha} - \beta & m\alpha & m \\ -\lambda(\overline{a})^{-1} & \lambda\overline{\alpha}(\overline{a})^{-1} & 0 & 0 \\ -\gamma\overline{y} + \delta\overline{x} & (\gamma\overline{y} - \delta\overline{x})\overline{\alpha} - \delta & \gamma\alpha & \gamma \end{pmatrix}.$$

If the above matrix lies in $\widetilde{U_p}$ then we have $p \mid \gamma\alpha$ which implies $p \mid \gamma$. But that immediately implies, by looking at the bottom left entry, that $p \mid \delta\overline{x}$, hence (by looking at the second entry of the bottom row) $p \mid \delta$. Thus $p \mid \gamma, p \mid \delta$, a contradiction.

Thus $\Theta h(l,0)t_i \notin P(\mathbb{Q}_p)\widetilde{U_p}$ if $i \in \{2,7\}$. For t_1 and t_5 we have the above decompositions, from which part (b) follows via the equations (5.2.1)-(5.2.7).

□

Proof of Theorem 5.3.1. By (3.5.1) we have

$$Z_p(s, W_p, B_p) = \sum_{l\geq 0, m\geq 0} B_p(h(l,m)) \sum_{t_i\in T_m} W_p(\Theta h(l,m)t_i, s) \cdot I_{t_i}^{l,m} \tag{5.3.3}$$

We first look at the terms corresponding to $m > 0$. From Lemma 5.3.2 and Proposition 3.4.1 we have $\sum_{t_i\in T_m} W_p(\Theta h(l,m)t_i, s) \cdot I_{t_i}^{l,m} = 0$. So only terms corresponding to $m = 0$ contribute.

From Proposition 3.4.2 and Lemma 5.3.2 we have

$$\sum_{t_i\in T_0} W_p(\Theta h(l,0)t_i, s) \cdot I_{t_i}^{l,0} = \frac{1}{p^2+1} \cdot p^{-3ls+l/2} a_p^l.$$

Hence (6.3.2) reduces to

$$Z_p(s, W_p, B_p) = \frac{1}{p^2+1} \cdot \sum_{l \geq 0} B_p(h(l,0)) p^{-3ls+l/2} a_p^l.$$

Define $C(y) = \sum_{l \geq 0} B_p(h(l,0)) y^l$. We are interested in the quantity

$$Z_p(s, W_p, B_p) = \frac{1}{p^2+1} C(a_p p^{-3s+1/2}). \tag{5.3.4}$$

Sugano, in [23, p. 544], has computed $C(y)$ explicitly. His results imply that

$$C(y) = \frac{H(y)}{Q(y)}$$

where $H(y) = 1 - \frac{y^2}{p^4}, Q(y) = \prod_{i=1}^4 (1 - \gamma_p^{(i)}(p) p^{-3/2} y)$.

Plugging in these values in (5.3.4) we get the desired result.

□

6. The case Steinberg π_p, Steinberg σ_p

6.1. **Assumptions.** Suppose that the characters $\omega_\pi, \omega_\sigma, \chi_0$ are trivial. Let $p \neq 2$ be a finite prime of $\mathbb{Q}$ such that

(a) p is inert in $L = \mathbb{Q}(\sqrt{-d})$.
(b) Λ_p is not trivial on $T(\mathbb{Z}_p)$; however it is trivial on $T(\mathbb{Z}_p) \cap \Gamma_p^0$.
(c) π_p is the Steinberg representation (or its twist by the unique non-trivial unramified quadratic character).
(d) σ_p is the Steinberg representation (or its twist by the unique non-trivial unramified quadratic character).
(e) The conductor of ψ_p is $\mathbb{Z}_p$.
(f) $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_p)$.
(g) $-d = b^2 - 4ac$ generates the discriminant of $L_p/\mathbb{Q}_p$.

Remark. π_p corresponds to a local newform for the Iwahori subgroup I_p (see [21]). Also, as in the previous section, σ_p corresponds to the local newform for the Iwahori subgroup $\Gamma_0(p)$ of $GL_2(\mathbb{Q}_p)$.

6.2. **Description of B_p and W_p.** Let Φ_p be the unique normalized local newform for the Iwahori subgroup I_p, as defined by Schmidt [21]. Let w_p be the local Atkin-Lehner eigenvalue for π_p; this equals -1 when π_p is the Steinberg representation and equals 1 when π_p is the unramified quadratic twist of the Steinberg representation. We let B_p be the normalized vector that corresponds to Φ_p in the Bessel space. Section 4 was devoted to the computation of the values $B_p(h(l,m)t)$ for $l, m \in \mathbb{Z}, m \geq 0, t \in T_m$.

Because p is inert, L_p is a quadratic extension of $\mathbb{Q}_p$ and we may write elements of L_p in the form $a + b\sqrt{-d}$ with $a, b \in \mathbb{Q}_p$; then $\mathbb{Z}_{L,p} = a + b\sqrt{-d}$ where $a, b \in \mathbb{Z}_p$. We also identify L_p with $T(\mathbb{Q}_p)$ and ξ with $\sqrt{-d}/2$. We now define W_p. By Assumption (b) above, we have Λ_p is trivial on the elements of $T(\mathbb{Q}_p)$ of the form $a + b\sqrt{-d}$ with $a, b \in \mathbb{Z}_p, p \mid b, p \nmid a$. Take the canonical map $r : \widetilde{K}_p \to \widetilde{G}(\mathbb{F}_p)$ and define $I_p' = r^{-1}(I(\mathbb{F}_p))$, where $I(\mathbb{F}_p)$ is the subgroup of $G(\mathbb{F}_p)$ defined in the beginning of this paper.

Let s_1 denote the matrix $\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$.

Let $W_p(\ ,s)$ be the unique vector in $I(\Pi_p, s)$ with the following properties:

- $W_p(1,s) = 1$,
- $W_p(s_1,s) = 1$,
- $W_p(gk,s) = W_p(g,s)$ is $k \in I'_p$,
- $W_p(g,s) = 0$ if g does not belong to $P(\mathbb{Q}_p)I'_p \sqcup P(\mathbb{Q}_p)s_1 I'_p$

Concretely we have the following description of $W_p(\ ,s)$:

We know that $\sigma_p = Sp \otimes \tau$ where Sp denotes the special (Steinberg) representation and τ is a (possibly trivial) unramified quadratic character. We put $a_p = \tau(p)$, thus $a_p = \pm 1$ is the eigenvalue of the local Hecke operator $T(p)$.

Let W'_p be the unique function on $GL_2(\mathbb{Q}_p)$ such that

$$W'_p(gk) = W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), k \in \Gamma_{0,p}, \tag{6.2.1}$$

$$W'_p(\begin{pmatrix}1 & x\\0 & 1\end{pmatrix} g) = \psi_p(-cx)W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), x \in \mathbb{Q}_p, \tag{6.2.2}$$

$$W'_p\begin{pmatrix}a & 0\\0 & 1\end{pmatrix} = \begin{cases}\tau(a)|a| & \text{if } |a|_p \leq 1,\\ 0 & \text{otherwise}\end{cases} \tag{6.2.3}$$

$$W'_p\left(\begin{pmatrix}a & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\right) = \begin{cases}-p^{-1}\tau(a)|a| & \text{if } |a|_p \leq p,\\ 0 & \text{otherwise}\end{cases} \tag{6.2.4}$$

We extend W'_p to a function on $GU(1,1)(\mathbb{Q}_p)$ by

$$W'_p(ag) = W'_p(g), \text{ for } a \in L_p^\times, g \in GL_2(\mathbb{Q}_p).$$

Then, $W_p(\ s)$ is the unique function on $\widetilde{G}(\mathbb{Q}_p)$ such that

$$W_p(mnuk,s) = W_p(mu,s), \text{ for } m \in M(\mathbb{Q}_p), n \in N(\mathbb{Q}_P), u \in \{1, s_1\}, k \in I'_p, \tag{6.2.5}$$

$$W_p(t) = 0 \text{ if } t \notin P(\mathbb{Q}_p)I'_p \sqcup P(\mathbb{Q}_p)s_1 I'_p \tag{6.2.6}$$

$$\begin{aligned} &W_p\left(\begin{pmatrix}a&0&0&0\\0&1&0&0\\0&0&\overline{a}^{-1}&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&a_1&0&b_1\\0&0&c_1&0\\0&d_1&0&e_1\end{pmatrix}u, s\right)\\ &= \left|N_{L/\mathbb{Q}}(a)\cdot c_1^{-1}\right|_p^{3(s+1/2)} \cdot \Lambda_p(a) W'_p\begin{pmatrix}a_1 & b_1\\ d_1 & e_1\end{pmatrix}, \end{aligned} \tag{6.2.7}$$

for $a \in \mathbb{Q}_p^\times, u \in \{1, s_1\}, \begin{pmatrix}a_1 & b_1\\ d_1 & e_1\end{pmatrix} \in GU(1,1)(\mathbb{Q}_p), c_1 = \mu_1\begin{pmatrix}a_1 & b_1\\ d_1 & e_1\end{pmatrix}$.

6.3. **The results.** We now state and prove the main theorem of this section.

Theorem 6.3.1. *Let the functions B_p, W_p be as defined in subsection 6.2. Then we have*

$$Z_p(s,W_p,B_p)=\frac{1-p}{p^2+1}\cdot\frac{p^{-6s-3}}{1-a_pw_pp^{-3s-3/2}}\cdot L(3s+\frac{1}{2},\pi_p\times\sigma_p)$$

where $L(s,\pi_p\times\sigma_p)=(1+a_pw_pp^{-1}p^{-s})^{-1}(1+a_pw_pp^{-2}p^{-s})^{-1}$.

Before we begin the proof, we need a lemma.

Lemma 6.3.2. *We have the following formulae for $W_p(\Theta h(l,m)t_i,s)$ where $t_i\in T_m$.*

$$W_p(\Theta h(l,m)t_i,s)=\begin{cases}p^{-6ms-3ls-3m-5l/2}a_p^l\cdot\frac{-1}{p} & \text{if } i=3,4, \quad m>0\\ p^{-6ms-3ls-3m-5l/2}a_p^l & \text{if } i=5,6, \quad m>0\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We have

$$\Theta h(l,m)=h(l,m)\begin{pmatrix}1&0&0&0\\ p^m\alpha&1&0&0\\ 0&0&1&-p^m\overline{\alpha}\\ 0&0&0&1\end{pmatrix} \tag{6.3.1}$$

Put $K_p'=r^{-1}(G(F_p))$. Thus $\Theta h(l,m)t_i\in P(\mathbb{Q}_p)K_p'$ when $m>0$ and $\Theta h(l,m)t_i\in P(\mathbb{Q}_p)\Theta K_p'$ when $m=0$. A direct computation shows that $P(\mathbb{Q}_p)K_p'$ and $P(\mathbb{Q}_p)\Theta K_p'$ are disjoint; the fact that $P(\mathbb{Q}_p)I_p'\subset P(\mathbb{Q}_p)K_p'$ then implies that $W_p(\Theta h(l,m)t_i,s)=0$ for $m=0$. From now on we assume $m>0$.

We can check that $\Theta h(l,m)t_i\notin P(\mathbb{Q}_p)I_p'\sqcup P(\mathbb{Q}_p)s_1I_p'$ if $i\in\{1,2,7,8\}$.

For the remaining t_i (i.e. $i\in\{3,4,5,6\}$) we have the decompositions:

$\Theta h(l,m)t_3=$

$$\begin{pmatrix}p^{2m+l}&0&0&0\\ 0&1&0&0\\ 0&0&p^{-2m-l}&0\\ 0&0&0&1\end{pmatrix}m^{(2)}(\begin{pmatrix}p^{m+l}&0\\ -p^m&p^m\end{pmatrix})\begin{pmatrix}-1&0&0&0\\ -p^m\alpha&-1&0&0\\ 0&-p^m\overline{\alpha}&-1&p^m\overline{\alpha}\\ -p^m\alpha&0&0&-1\end{pmatrix}$$

$\Theta h(l,m)t_4=$

$$\begin{pmatrix}p^{2m+l}&0&0&0\\ 0&1&0&0\\ 0&0&p^{-2m-l}&0\\ 0&0&0&1\end{pmatrix}m^{(2)}(\begin{pmatrix}p^{m+l}&0\\ -p^m&p^m\end{pmatrix})s_1\begin{pmatrix}1&p^m\alpha&0&0\\ 0&1&0&0\\ 0&p^m\alpha&1&0\\ p^m\overline{\alpha}&0&-p^m\overline{\alpha}&1\end{pmatrix}$$

$\Theta h(l,m)t_5$

$$=\begin{pmatrix}p^{2m+l}&0&0&0\\ 0&1&0&0\\ 0&0&p^{-2m-l}&0\\ 0&0&0&1\end{pmatrix}m^{(2)}(\begin{pmatrix}p^{m+l}&0\\ 0&p^m\end{pmatrix})\begin{pmatrix}1&0&0&0\\ p^m\alpha&1&0&0\\ 0&0&1&p^m\overline{\alpha}\\ 0&0&0&1\end{pmatrix}$$

$\Theta h(l,m)t_6$

$$= \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ 0 & p^m \end{pmatrix}) s_1 \begin{pmatrix} 1 & p^m\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -p^m\overline{\alpha} & 1 \end{pmatrix}$$

The lemma now follows from the above decompositions and equations (6.2.1)-(6.2.7).

□

Proof of Theorem 6.3.1. By (3.5.1) we have

$$Z_p(s, W_p, B_p) = \sum_{l\geq 0, m\geq 0} \sum_{t_i \in T_m} B_p(h(l,m)t_i) W_p(\Theta h(l,m)t_i, s) \cdot I_{t_i}^{l,m} \tag{6.3.2}$$

From Proposition 3.4.1, Proposition 4.3.1 and Lemma 6.3.2 we have

$$\sum_{i\in\{3,4,5,6\}} B_p(h(l,m)t_i) W_p(\Theta h(l,m)t_i, s) \cdot I_{t_i}^{l,m} = \frac{(1-p)(-a_p w_p p^{-3s-5/2})^l (p^{-6s-3})^m}{p^2+1}.$$

Hence (6.3.2) implies

$$Z_p(s, W_p, B_p) = \frac{(1-p)p^{-6s-3}}{p^2+1} \cdot \frac{1}{1+a_p w_p p^{-2} p^{-3s-1/2}} \cdot \frac{1}{1-p^{-6s-3}}$$

This completes the proof.

□

Remark. We might equally well have chosen W_p to be the simpler vector supported only on 1 (rather than on 1 and s_1). Indeed, all the results in this paper will remain valid with that choice. The reason we include s_1 in the support of the section is because this definition will be necessary for our future work [20].

7. The case Steinberg π_p, unramified σ_p

7.1. **Assumptions.** Suppose that the characters $\omega_\pi, \omega_\sigma, \chi_0$ are trivial. Let $p \neq 2$ be a finite prime of $\mathbb{Q}$ such that

(a) p is inert in $L = \mathbb{Q}(\sqrt{-d})$.
(b) Λ_p is not trivial on $T(\mathbb{Z}_p)$; however it is trivial on $T(\mathbb{Z}_p) \cap \Gamma_p^0$.
(c) π_p is the Steinberg representation (or its twist by the unique non-trivial unramified quadratic character) while σ_p is unramified.
(d) The conductor of ψ_p is $\mathbb{Z}_p$.
(e) $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_p)$.
(f) $-d = b^2 - 4ac$ generates the discriminant of $L_p/\mathbb{Q}_p$.

Remark. π_p corresponds to a local newform for the Iwahori subgroup I_p (see [21]).

7.2. **Description of B_p and W_p.** Let Φ_p be the unique normalized local newform for the Iwahori subgroup I_p, as defined by Schmidt [21]. Let w_p be the local Atkin-Lehner eigenvalue for π_p; this equals -1 when π_p is the Steinberg representation and equals 1 when π_p is the unramified quadratic twist of the Steinberg representation. We let B_p be the normalized vector that corresponds to Φ_p in the Bessel space. Section 4 was devoted to the computation of the values $B_p(h(l,m)t)$ for $l, m \in \mathbb{Z}, m \geq 0, t \in T_m$.

We now define W_p. Take the canonical map $r : \widetilde{K}_p \to \widetilde{G}(\mathbb{F}_p)$ and define $I'_p = r^{-1}(I(\mathbb{F}_p))$, where $I(\mathbb{F}_p)$ is the subgroup of $G(\mathbb{F}_p)$ defined in the beginning of this paper.

Let $W_p(\ ,s)$ be the unique vector in $I(\Pi_p, s)$ with the following properties:

- $W_p(\Theta, s) = 1$,
- $W_p(1, s) = 1$,
- $W_p(gk, s) = W_p(g, s)$ is $k \in I'_p$,
- $W_p(g, s) = 0$ if g does not belong to $P(\mathbb{Q}_p)\Theta I'_p \sqcup P(\mathbb{Q}_p)I'_p$

Concretely we have the following description of $W_p(\ ,s)$.

Suppose σ_p is the principal series representation induced from the unramified characters α, β of $\mathbb{Q}_p^\times$. Let W'_p be the unique function on $GL_2(\mathbb{Q}_p)$ such that

$$W'_p(gk) = W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), k \in GL_2(\mathbb{Z}_p), \tag{7.2.1}$$

$$W'_p(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \psi_p(-cx)W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), x \in \mathbb{Q}_p, \tag{7.2.2}$$

$$W'_p\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{cases} \left|\frac{a}{b}\right|_p^{\frac{1}{2}} \cdot \frac{\alpha(ap)\beta(b)-\alpha(b)\beta(ap)}{\alpha(p)-\beta(p)} & \text{if } \left|\frac{a}{b}\right|_p \leq 1, \\ 0 & \text{otherwise} \end{cases} \tag{7.2.3}$$

We extend W'_p to a function on $GU(1,1)(\mathbb{Q}_p)$ by

$$W'_p(ag) = W'_p(g), \text{ for } a \in L_p^\times, g \in GL_2(\mathbb{Q}_p).$$

Then, $W_p(\,s)$ is the unique function on $\widetilde{G}(\mathbb{Q}_p)$ such that

(7.2.4)
$$W_p(mnuk, s) = W_p(mu, s), \text{ for } m \in M(\mathbb{Q}_p), n \in N(\mathbb{Q}_P), u \in \{1, \Theta\}, k \in I'_p,$$

$$W_p(t) = 0 \text{ if } t \notin P(\mathbb{Q}_p)\Theta I'_p \sqcup P(\mathbb{Q}_p)I'_p \tag{7.2.5}$$

$$W_p\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & b_1 \\ 0 & 0 & c_1 & 0 \\ 0 & d_1 & 0 & e_1 \end{pmatrix} u, s\right) = \left|N_{L/\mathbb{Q}}(a) \cdot c_1^{-1}\right|_p^{3(s+1/2)} \cdot \Lambda_p(\overline{a}^{-1}) W'_p\begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix}, \tag{7.2.6}$$

for $a \in \mathbb{Q}_p^\times, u \in \{1, \Theta\}, \begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix} \in GU(1,1)(\mathbb{Q}_p), c_1 = \mu_1\begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix}$.

7.3. **The results.** We now state and prove the main theorem of this section.

Theorem 7.3.1. *Let the functions B_p, W_p be as defined in subsection 7.2. Then we have*

$$Z_p(s, W_p, B_p) = \frac{1}{(p+1)(p^2+1)} \cdot L(3s + \frac{1}{2}, \pi_p \times \sigma_p),$$

where $L(s, \pi_p \times \sigma_p) = (1 + w_p p^{-3/2}\alpha(p)p^{-s})^{-1}(1 + w_p p^{-3/2}\beta(p)p^{-s})^{-1}$.

Before we begin the proof, we need a lemma.

Lemma 7.3.2. *Let $t_i \in T_m, l \geq 0$. We have*

$$W_p(\Theta h(l,m)t_i, s) = \begin{cases} p^{-3ls-2l}\left(\frac{\alpha(p)^{l+1}-\beta(p)^{l+1}}{\alpha(p)-\beta(p)}\right) & \text{if } m = 0, i = 5 \\ p^{-6ms-3ls-3m-5l/2}\left(\frac{\alpha(p)^{l+1}-\beta(p)^{l+1}}{\alpha(p)-\beta(p)}\right) & \text{if } m > 0, i = 3,5 \\ 0 & \text{otherwise} \end{cases}$$

Proof. By the proof of Lemma 6.3.2 we have $\Theta h(l,m)t_i \notin P(\mathbb{Q}_p)\Theta I'_p$ if $m > 0$. As for the case $m = 0$, we can check that $\Theta h(l,0)t_i \notin P(\mathbb{Q}_p)\Theta I'_p$ if $i \in \{1,2,7\}$. On the other hand, again by the proof of Lemma 6.3.2, we have $\Theta h(l,m)t_i \in P(\mathbb{Q}_p)I'_p$ if and only if $m > 0$ and $i \in \{3,5\}$. The lemma now follows immediately from (7.2.1) - (7.2.6). □

Proof of Theorem 7.3.1. We have

$$\begin{aligned} Z_p(s, W_p, B_p) &= \sum_{l\geq 0} W_p(\Theta h(l,0)t_5, s)B_p(h(l,0)t_5) \cdot I^{l,0}_{t_5} \\ &\quad + \sum_{l\geq 0, m>0} \sum_{i\in\{3,5\}} W_p(\Theta h(l,m)t_i, s)B_p(h(l,m)t_i) \cdot I^{l,m}_{t_i} \end{aligned} \tag{7.3.1}$$

Using Proposition 4.3.2 , Proposition 3.4.2 and Lemma 7.3.2 we have

$$\sum_{i\in\{3,5\}} W_p(\Theta h(l,m)t_i, s)B_p(h(l,m)t_i) \cdot I^{l,m}_{t_i} = 0$$

and hence

$$\begin{aligned} Z_p(s, W_p, B_p) &= \sum_{l\geq 0} W_p(\Theta h(l,0)t_5, s)B_p(h(l,0)t_5) \cdot I^{l,0}_{t_5} \\ &= \frac{1}{(p+1)(p^2+1)} \sum_{l\geq 0} p^{-3ls-2l}\left(\frac{\alpha(p)^{l+1}-\beta(p)^{l+1}}{\alpha(p)-\beta(p)}\right)(-pw_p)^l p^{-l} \\ &= \frac{1}{(p+1)(p^2+1)} L(3s + \frac{1}{2}, \pi_p \times \sigma_p). \end{aligned}$$

This completes the proof of the theorem.

□

Remark. We might equally well have chosen W_p to be the simpler vector supported only on Θ (rather than on Θ and 1). The only reason we include 1 in the support of the section is because this definition will be necessary for our future work [20].

8. The global integral and some results

8.1. **Classical Siegel modular forms and newforms for the minimal congruence subgroup.** For M a positive integer define the following global parahoric subgroups.

$$B(M) := Sp(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_1(M) := Sp(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_2(M) := Sp(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_0(M) := Sp(4,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & M^{-1}\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

When $M = 1$ each of the above groups is simply $Sp(4,\mathbb{Z})$. For $M > 1$, the groups are all distinct. If Γ' is equal to one of the above groups, or (more generally) is any congruence subgroup, we define $S_k(\Gamma')$ to be the space of Siegel cusp forms of degree 2 and weight k with respect to the group Γ'.

More precisely, let $\mathbb{H}_2 = \{Z \in M_2(\mathbb{C}) | Z = Z^T, i(\overline{Z} - Z) \text{ is positive definite}\}$. For any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ let $J(g,Z) = CZ + D$. Then $f \in S_k(\Gamma')$ if it is a holomorphic function on $\mathbb{H}_2$, satisfies $f(\gamma Z) = \det(J(\gamma, Z))^k f(Z)$ for $\gamma \in \Gamma', Z \in \mathbb{H}_2$ and disappears at the cusps. This last condition can be rephrased as follows. f has a Fourier expansion

$$f(Z) = \sum_{S>0} a(S,F) e(\mathrm{tr}(SZ)),$$

where $e(z) = \exp(2\pi i z)$ and S runs through all symmetric semi-integral positive-definite matrices of size two.

Now let M be a square-free positive integer. For any decomposition $M = M_1M_2$ into coprime integers we define, following Schmidt [21], the subspace of oldforms $S_k(B(M))^{\mathrm{old}}$ to be the sum of the spaces

$$S_k(B(M_1) \cap U_0(M_2)) + S_k(B(M_1) \cap U_1(M_2)) + S_k(B(M_1) \cap U_2(M_2)).$$

For each prime p not dividing M there is the local Hecke algebra $\mathfrak{H}_p$ of operators on $S_k(B(M))$ and for each prime q dividing M we have the Atkin-Lehner involution η_q also acting on $S_k(B(M))$. For details, the reader may refer to [21].

By a newform for the minimal congruence subgroup $B(M)$, we mean an element $f \in S_k(B(M))$ with the following properties

(a) f lies in the orthogonal complement of the space $S_k(B(M))^{\mathrm{old}}$.
(b) f is an eigenform for the local Hecke algebras $\mathfrak{H}_p$ for all primes p not dividing M.
(c) f is an eigenform for the Atkin-Lehner involutions η_q for all primes q dividing M.

Remark. By [21], if we assume the hypothesis that a nice L-function theory for $GSp(4)$ exists, (b) and (c) above follow from (a) and the assumption that f is an eigenform for the local Hecke algebras at *almost* all primes.

8.2. **Description of our newforms.** Let M be an odd square-free positive integer and

$$F(Z) = \sum_{T>0} a(T)\mathrm{e}(\mathrm{tr}(TZ))$$

be a Siegel newform for $B(M)$ of even weight l.

Let N be an odd square-free positive integer and g be a normalized newform of weight l for $\Gamma_0(N)$. g has a Fourier expansion

$$g(z) = \sum_{n=1}^{\infty} b(n)e(nz)$$

with $b(1) = 1$. It is then well known that the $b(n)$ are all totally real algebraic numbers.

We make the following assumption:

$$a(T) \neq 0 \textit{ for some } T = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \tag{8.2.1}$$

such that $-d = b^2 - 4ac$ *is the discriminant of the imaginary quadratic field* $\mathbb{Q}(\sqrt{-d})$*, and all primes dividing* MN *are inert in* $\mathbb{Q}(\sqrt{-d})$.

We define a function $\Phi = \Phi_F$ on $G(\mathbb{A})$ by

$$\Phi(\gamma h_\infty k_0) = \mu_2(h_\infty)^l \det(J(h_\infty, iI_2))^{-l} F(h_\infty(i))$$

where $\gamma \in G(\mathbb{Q}), h_\infty \in G(\mathbb{R})^+$ and

$$k_0 \in (\prod_{p\nmid M} K_p) \cdot (\prod_{p|M} I_p).$$

Because we do not have strong multiplicity one for G we can only say that the representation of $G(\mathbb{A})$ generated by Φ is a *multiple* of an irreducible representation π. However that is enough for our purposes.

We know that $\pi = \otimes \pi_v$ where

$$\pi_v = \begin{cases} \text{holomorphic discrete series} & \text{if } v = \infty, \\ \text{unramified spherical principal series} & \text{if } v \text{ finite }, v \nmid M, \\ \xi_v \mathrm{St}_{GSp(4)} \text{where } \xi_v \text{ unramified, } \xi_v^2 = 1 & \text{if } v \mid M. \end{cases}$$

Next, we define a function Ψ on $GL_2(\mathbb{A})$ by

$$\Psi(\gamma_0 m k_0) = (\det\ m)^{\frac{l}{2}} (\gamma i + \delta)^{-l} g(m(i))$$

where $\gamma_0 \in GL_2(\mathbb{Q})$, $m = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2^+(\mathbb{R})$, and

$$k_0 \in \prod_{p \nmid N} GL_2(\mathbb{Z}_p) \prod_{p|N} \Gamma_{0,p}$$

Let σ be the automorphic representation of $GL_2(\mathbb{A})$ generated by Ψ.

We know that $\sigma = \otimes \sigma_v$ where

$$\sigma_v = \begin{cases} \text{holomorphic discrete series} & \text{if } v = \infty, \\ \text{unramified spherical principal series} & \text{if } v \text{ finite }, v \nmid N, \\ \xi \mathrm{St}_{GL(2)} \text{where } \xi_v \text{ unramified, } \xi_v^2 = 1 & \text{if } v \mid N. \end{cases}$$

8.3. **Description of our Bessel model.** In order to use our results from the previous sections, we need to associate a Bessel model to π (or more accurately, we associate it to $\widetilde{\pi}$). This involves making a choice of (S, Λ, ψ). This subsection is devoted to doing that.

Let $\psi = \prod_v \psi_v$ be a character of $\mathbb{A}$ such that

- The conductor of ψ_p is $\mathbb{Z}_p$ for all (finite) primes p,
- $\psi_\infty(x) = e(-x)$, for $x \in \mathbb{R}$,
- $\psi|_\mathbb{Q} = 1$.

Put $L = \mathbb{Q}(\sqrt{-d})$. where d is the integer defined in (8.2.1).

First we deal with the case $M = 1$. In this case, our choice of S and Λ is identical to [2]. To recall, put

$$T(\mathbb{A}) = \coprod_{j=1}^{h(-d)} t_j T(\mathbb{Q}) T(\mathbb{R}) (\Pi_{p<\infty} T(\mathbb{Z}_p)) \tag{8.3.1}$$

where $t_j \in \prod_{p<\infty} T(\mathbb{Q}_p)$ and $h(-d)$ is the class number of L.

Write $t_j = \gamma_j m_j \kappa_j$, where $\gamma_j \in GL_2(\mathbb{Q}), m_j \in GL_2^+(\mathbb{R})$, and $\kappa_j \in ((\Pi_{p<\infty} GL_2(\mathbb{Z}_p))$. Choose

$$S = \begin{cases} \begin{pmatrix} d/4 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } d \equiv 0 \pmod 4 \\ \begin{pmatrix} (1+d)/4 & 1/2 \\ 1/2 & 1 \end{pmatrix} & \text{if } d \equiv 3 \pmod 4 \end{cases}$$

Let $S_j = \det(\gamma_j)^{-1} \gamma_j^T S \gamma_j$. Then, any primitive semi-integral two by two positive definite matrix with discriminant equal to $-d$ is $SL_2(\mathbb{Z})$-equivalent to some S_j. So, by our assumption, we can choose Λ a character of $T(\mathbb{A})/T(\mathbb{Q})T(\mathbb{R})((\Pi_{p<\infty} T(\mathbb{Z}_p))$ such that

$$\sum_{j=1}^{h(-d)} \Lambda(t_j) \overline{a(S_j)} \neq 0.$$

Thus, we have specified a choice of S and Λ for $M = 1$.

In the rest of this subsection, unless otherwise mentioned, assume $M > 1$.

Suppose p is a prime dividing M. We can identify L_p with elements $a + b\sqrt{-d}$ with $a, b \in \mathbb{Q}_p$. Let $\mathbb{Z}_{L,p}^\times$ denote the units in the ring of integers of L_p. The elements of $\mathbb{Z}_{L,p}^\times$ are of the form $a + b\sqrt{-d}$ with $a, b \in \mathbb{Z}_p$ and such that at least one of a and b is a unit. Let $\Gamma_{L,p}^0$ be the subgroup of $\mathbb{Z}_{L,p}^\times$ consisting of the elements with $p|b$. The group $\mathbb{Z}_{L,p}^\times / \Gamma_{L,p}^0$ is clearly cyclic of order $p + 1$. Moreover, the elements

$\{(-b+\sqrt{-d})/2\}$ where b is a positive integer satisfying $\{1 \le b \le 2p : b = d \pmod 2\}$ are distinct in $\mathbb{Z}_{L,p}^{\times}/\Gamma_{L,p}^{0}$. Note that $d = 0$ or $3 \pmod 4$ and hence $b = d \pmod 2$ implies that 4 divides $b^2 + d$. So we have the lemma:

Lemma 8.3.1. *There exists an integer b such that 4 divides b^2+d and $(-b+\sqrt{-d})/2$ is a generator of the group $\mathbb{Z}_{L,p}^{\times}/\Gamma_{L,p}^{0}$ for each $p|M$.*

Proof. By the comments above, we can choose, for each prime p_i dividing M, an integer b_i such that $b_i \equiv d \pmod 2$ and $(-b_i+\sqrt{-d})/2$ is a generator of the group $\mathbb{Z}_{L,p_i}^{\times}/\Gamma_{L,p_i}^{0}$. Now, using the Chinese Remainder theorem, choose b satisfying $b \equiv b_i \pmod{2p_i}$ for each i . □

Now we define

$$S = \begin{pmatrix} \frac{b^2+d}{4} & \frac{b}{2} \\ \frac{b}{2} & 1 \end{pmatrix}.$$

As in section 1.1 we define the matrix $\xi = \xi_S$ and the group $T = T_S$. We have $T(\mathbb{Q}) \simeq L^{\times}$. We write $T(\mathbb{Z}_p)$ for $T(\mathbb{Q}_p) \cap GL_2(\mathbb{Z}_p)$.

Let

$$T(\mathbb{A}) = \coprod_{j=1}^{h(-d)} t_j T(\mathbb{Q}) T(\mathbb{R}) (\Pi_{p<\infty} T(\mathbb{Z}_p) \tag{8.3.2}$$

where $t_j \in \prod_{p<\infty} T(\mathbb{Q}_P)$ and $h(-d)$ is the class number of L. For each $p|M$ put $\Gamma_{L,p}^{0} = T(\mathbb{Z}_p) \cap \Gamma_p^0$. Note that under the isomorphism $T(\mathbb{Z}_p) \simeq \mathbb{Z}_{L,p}^{\times}$ sending $x + y\xi \mapsto x + y\frac{\sqrt{-d}}{2}$, our two definitions for $\Gamma_{L,p}^{0}$ agree, so there is no ambiguity.

Let $M = p_1 p_2 ... p_r$ be its decomposition into distinct primes. For each $1 \le i \le r$ we choose coset representatives $u_{k_i}^{(p_i)} \in T(\mathbb{Z}_{p_i})$ such that

$$T(\mathbb{Z}_{p_i}) = \coprod_{k_i=1}^{p_i+1} u_{k_i}^{(p_i)} \Gamma_{L,p_i}^{0}.$$

We write an r-tuple $(k_1, .., k_r)$ in short as $\widetilde{k}$. Let X denote the cartesian product of the r sets $X_i = \{x : 1 \le x \le p_i\}$. For $\widetilde{k} \in X$, define

$$u_{\widetilde{k}} = \prod_{i=1}^{r} u_{k_i}^{(p_i)}.$$

Then it is easy to see that as $\widetilde{k}$ varies over X the elements $u_{\widetilde{k}}$ form a set of coset representatives of $\Pi_{p|M} T(\mathbb{Z}_p)/\Pi_{p|M}\Gamma_{L,p}^{0}$. Also note that $|X| = |SL_2(\mathbb{Z})/\Gamma^0(M)| = \Pi_{p_1|M}(p_i+1)$. We denote the quantity $\Pi_{p_1|M}(p_i+1)$ by $g(M)$.

Let $T(\mathbb{Z})$ denote the (finite) group of units in the ring of integers $\mathbb{Z}_L$ of L. Let $t(d)$ denote the cardinality of the group $T(\mathbb{Z})/\{\pm 1\}$. We know that,

$$t(d) = \begin{cases} 3 & \text{if } d = 3 \\ 2 & \text{if } d = 4 \\ 1 & \text{otherwise.} \end{cases}$$

Let $T_M^{\times}$ be the image of $T(\mathbb{Z})$ in $\Pi_{p|M} T(\mathbb{Z}_p)$. Then $T_M^{\times} \cap \Pi_{p|M}\Gamma_{L,p}^{0} = \{\pm 1\}$. Choose a set of elements $r_1, r_2, ..r_{t(d)}$ in $T(\mathbb{Z})$ such that they form distinct representatives in $T(\mathbb{Z})/\{\pm 1\}$. Let $\overline{r}_i$ denote the image of r_i in $T_M^\times$. We have

$$T_M^\times \Pi_{p|M} \Gamma^0_{L,p} = \coprod_{i=1}^{t(d)} \overline{r}_i (\Pi_{p|M} \Gamma^0_{L,p}). \tag{8.3.3}$$

Finally, choose $x_1, x_2, ..., x_{g(M)/t(d)}$ in $\Pi_{p|M} T(\mathbb{Z}_p)$ such that we have the disjoint coset decomposition:

$$\Pi_{p|M} T(\mathbb{Z}_p) = \coprod_{i=1}^{g(M)/t(d)} x_i T_M^\times \Pi_{p|M} \Gamma^0_{L,p} \tag{8.3.4}$$

This immediately gives us the fundamental coset decomposition:

$$T(\mathbb{A}) = \coprod_{\substack{1 \le j \le h(-d) \\ 1 \le k \le g(M)/t(d)}} t_j x_k T(\mathbb{Q}) T(\mathbb{R}) (\Pi_{p \nmid M} T(\mathbb{Z}_p)) (\Pi_{p_i | M} \Gamma^0_{L,p_i}) \tag{8.3.5}$$

Also from (8.3.3) and (8.3.4) we immediately get another coset decomposition:

$$\Pi_{p|M} T(\mathbb{Z}_p) = \coprod_{\substack{1 \le i \le g(M)/t(d) \\ 1 \le j \le t(d)}} x_i \overline{r}_j \Pi_{p|M} \Gamma^0_{L,p} \tag{8.3.6}$$

But we know that an alternate set of coset representatives in the above equation is given by the elements $u_{\widetilde{k}}$. It follows that for any $1 \le i \le g(M)/t(d), 1 \le j \le t(d)$, there exists a unique $\widetilde{k} \in X$ such that $u_{\widetilde{k}}^{-1} x_i \overline{r}_j \in \Pi_{p|M} \Gamma^0_{L,p}$. This correspondence is bijective.

Write $t_j x_k = \gamma_{j,k} m_{j,k} \kappa_{j,k}$, where $\gamma_{j,k} \in GL_2(\mathbb{Q}), m_{j,k} \in GL_2^+(\mathbb{R})$, and $\kappa_{j,k} \in (\Pi_{p<\infty, p \nmid M} GL_2(\mathbb{Z}_p) \cdot \Pi_{p|M} \Gamma^0_p$. Also, by $(\gamma_{j,k})_f$ we denote the finite part of $\gamma_{j,k}$, that is, $(\gamma_{j,k})_f = \gamma_{j,k} m_{j,k}$.

Lemma 8.3.2. *For each j, the elements $\gamma_{j,1}^{-1} r_l \gamma_{j,k}$ form a system of representatives of $SL_2(\mathbb{Z})/\Gamma^0(M)$ as l, k vary over $1 \le l \le t(d)$, $1 \le k \le g(M)/t(d)$.*

Proof. Fix j. Let $1 \le l_2 \le t(d)$, $1 \le k_2 \le g(M)/t(d)$. We have

$$\gamma_{j,k_2}^{-1} r_{l_2}^{-1} r_l \gamma_{j,k} = m_{j,k_2} \kappa_{j,k_2} x_{k_2}^{-1} r_{l_2}^{-1} r_l x_k (m_{j,k} \kappa_{j,k})^{-1}.$$

Therefore $\gamma_{j,k_2}^{-1} r_{l_2}^{-1} r_l \gamma_{j,k} \in \left(GL_2^+(\mathbb{R}) \Pi_{q<\infty} GL_2(\mathbb{Z}_q)\right) \cap GL_2(\mathbb{Q}) = SL_2(\mathbb{Z})$. Moreover, if it belongs to $\Gamma^0(M)$ then we must have $x_{k_2}^{-1} \overline{r}_{l_2}^{-1} \overline{r}_l x_k \in \Pi_{p|M} \Gamma^0_p$ and by (8.3.6) this can happen only if $l = l_2, k = k_2$. Now the lemma follows because the size of the set $\gamma_{j,1}^{-1} r_l \gamma_{j,k}$ equals the cardinality of $SL_2(Z)/\Gamma^0(M)$. □

Let $S_{j,k} = \det(\gamma_{j,k})^{-1} \gamma_{j,k}^T S \gamma_{j,k}$. So, looking at S and $S_{j,k}$ as elements of $GL_2(R)^+$ we have $S_{j,k} = \det(m_{j,k})\ (m_{j,k}^{-1})^T S m_{j,k}^{-1}$.

Lemma 8.3.3. *There exists j, k, $1 \le j \le h(-d)$, $1 \le k \le g(M)/t(d)$ such that $a(S_{j,k}) \neq 0$.*

Proof. By assumption (8.2.1), $a(T) \neq 0$ for some primitive semi-integral positive definite matrix T with discriminant equal to $-d$. By [2, p.209] there exists j such that T is $SL_2(\mathbb{Z})$-equivalent to $S_{j,1}$. This means there is $R \in SL_2(\mathbb{Z})$ such that

$T = R^T S_{j,1} R$. By Lemma 8.3.2, we can find k, l such that $R = \gamma_{j,1}^{-1} r_l \gamma_{j,k} g$ where $g \in \Gamma^0(M)$. This gives us

$$\begin{aligned} T &= g^T \gamma_{j,k}^T r_l^T (\gamma_{j,1}^{-1})^T S_{j,1} \gamma_{j,1}^{-1} r_l \gamma_{j,k} g \\ &= \det(\gamma_{j,k})^{-1} g^T \gamma_{j,k}^T r_l^T S r_l \gamma_{j,k} g \\ &= \det(\gamma_{j,k})^{-1} g^T \gamma_{j,k}^T S \gamma_{j,k} g \\ &= g^T S_{j,k} g \end{aligned}$$

Hence $0 \neq a(T) = a(g^T S_{j,k} g) = a(S_{j,k})$, using the fact that the image of g^T in $Sp_4(\mathbb{Z})$ falls in $B(M)$ and F is a modular form for $B(M)$.

□

Proposition 8.3.4. *There exists a character* Λ *of* $T(\mathbb{A})/(T(\mathbb{Q})T(\mathbb{R})\Pi_{p<\infty, p\nmid M} T(\mathbb{Z}_p) \cdot \Pi_{p|M} \Gamma_{L,p}^0)$ *such that*

$$\sum_{\substack{1 \le j \le h(-d) \\ 1 \le k \le g(M)/t(d)}} \Lambda(t_j x_k)^{-1} \overline{a(S_{j,k})} \neq 0.$$

Moreover for any such Λ *we have* Λ_p *non-trivial on* $T(\mathbb{Z}_p)$ *for each prime* $p|M$.

Proof. By Lemma 8.3.3 we can find $S_{j,k}$ such that $a(S_{j,k}) \neq 0$. Hence using (8.3.5) we know that a character Λ satisfying the condition listed in the proposition exists.

Let Λ be such a character and p_i a fixed prime dividing M. We will show that Λ_{p_i} is not the trivial character on $T(\mathbb{Z}_{p_i})$.

For any $1 \le j \le h(-d)$ and $\widetilde{k} \in X$ we can write $t_j u_{\widetilde{k}} = \gamma_{j,\widetilde{k}} m_{j,\widetilde{k}} \kappa_{j,\widetilde{k}}$, where $\gamma_{j,\widetilde{k}} \in GL_2(\mathbb{Q}), m_{j,\widetilde{k}} \in GL_2^+(\mathbb{R})$ and $\kappa_{j,k} \in (\Pi_{p<\infty, p\nmid M} GL_2(\mathbb{Z}_p) \cdot \Pi_{p|M} \Gamma_p^0$.

We put $S_{j,\widetilde{k}} = \det(\gamma_{j,\widetilde{k}})^{-1} \gamma_{j,\widetilde{k}}^T S \gamma_{j,k}$

Suppose Λ_{p_i} is trivial on $T(\mathbb{Z}_{p_i})$. We claim that

$$\sum_{\substack{1 \le j \le h(-d) \\ \widetilde{k} \in X}} \Lambda(t_j u_{\widetilde{k}})^{-1} \overline{a(S_{j,\widetilde{k}})} = 0. \tag{8.3.7}$$

Suppose we fix $k_1, k_2, .., k_{i-1}, k_{i+1}, ..k_r$. For $1 \le y \le p_i + 1$, let $\widetilde{k}^y \in X$ be the r-tuple obtained by putting $k_i = y$. Then, by essentially the same argument as in Lemma 8.3.2 we see that $\gamma_{j,\widetilde{k}^1}^{-1} \gamma_{j,\widetilde{k}^y}$ form a set of representatives of $\Gamma^0(M/p_i)/\Gamma^0(M)$. In particular, this implies, by [21, 3.3.3], that $\sum_y a(S_{j,\widetilde{k}^y}) = 0$, and therefore, because Λ_{p_i} is trivial on $T(\mathbb{Z}_{p_i})$, we must have $\sum_y \Lambda(t_j u_{\widetilde{k}^y})^{-1} a(S_{j,\widetilde{k}^y}) = 0$. It follows, by breaking up

$$\sum_{\substack{1 \le j \le h(-d) \\ \widetilde{k} \in X}} \Lambda(t_j u_{\widetilde{k}})^{-1} \overline{a(S_{j,\widetilde{k}})}$$

into quantities as above, (8.3.7) follows.

Given $1 \le k \le g(M)/t(d), 1 \le l \le t(d)$, let $\widetilde{k}(k,l)$ be the unique element in X such that

$$u_{\widetilde{k}(k,l)}^{-1} x_k \overline{r}_l \in \Pi_{p|M} \Gamma_{L,p}^0 \tag{8.3.8}$$

. Such an element exists by our comment after (8.3.6). Suppose we write $r_l = \overline{r}_l r_{l,f} r_{l,\infty}$ where $r_{l,f} \in \Pi_{p\nmid M} T(\mathbb{Z}_p)$ and $r_{l,\infty} \in T(\mathbb{R})$

Then, using (8.3.8) we have

$$t_j u_{\widetilde{k}(k,l)} = r_l t_j x_k r_{l,\infty}^{-1} k$$

with $k \in (\Pi_{p<\infty, p\nmid M} GL_2(\mathbb{Z}_p)\cdot \Pi_{p|M}\Gamma_p^0$. In other words we can take $\gamma_{j,\widetilde{k}(k,l)} = r_l\gamma_{j,k}$.

But then $a(S_{j,\widetilde{k}(k,l)}) = a(S_{j,k})$. Also from (8.3.8) it is clear that $\Lambda^{-1}(t_j u_{\widetilde{k}(k,l)}) = \Lambda^{-1}(t_j x_k)$. On the other hand if we let k,l vary over all elements in the range $1\le k\le g(M)/t(d), 1\le l\le t(d)$, the corresponding $\widetilde{k}(k,l)$ vary over all $\widetilde{k}\in X$. As a result we conclude that

$$\sum_{\substack{1\le j\le h(-d)\\ \widetilde{k}\in X}} \Lambda(t_j u_{\widetilde{k}})^{-1}\overline{a(S_{j,\widetilde{k}})} = t(d)\sum_{\substack{1\le j\le h(-d)\\ 1\le k\le g(M)/t(d)}} \Lambda(t_j x_k)^{-1}\overline{a(S_{j,k})} \tag{8.3.9}$$

But we have already shown that if Λ_{p_i} is trivial on $T(\mathbb{Z}_{p_i})$ then

$$\sum_{\substack{1\le j\le h(-d)\\ \widetilde{k}\in X}} \Lambda(t_j u_{\widetilde{k}})^{-1}\overline{a(S_{j,\widetilde{k}})} = 0.$$

The proof follows.

□

Consider now the global Bessel space of type (S,Λ,ψ) for $\widetilde{\pi}$. We shall prove that this space is non zero.

For that, we consider

$$B_{\overline{\Phi}}(h) = \int_{Z_G(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})} (\Lambda\otimes\theta)(r)^{-1}\overline{\Phi}(rh)dr \tag{8.3.10}$$

where θ is defined as in Section 1 and $\overline{\Phi}(h) = \overline{\Phi(h)}$. We will show that this function is non-zero. In fact, we shall explicitly evaluate $B_{\overline{\Phi}}(g_\infty)$ for $g_\infty\in G(\mathbb{R})^+$.

Proposition 8.3.5. *Let $g_\infty \in G(\mathbb{R})^+$ and define $B_{\overline{\Phi}}(g_\infty)$ as in (8.3.10). The following hold:*

(a) *If $M=1$ we have*

$$B_{\overline{\Phi}}(g_\infty) = \overline{\det(J(g_\infty,i))^{-l}}\mu_2(g_\infty)^l e(-tr(S\cdot\overline{g_\infty(i)}) \sum_{1\le j\le h(-d)}\Lambda(t_j)^{-1}\overline{a(S_j)}$$

(b) *If $M>1$ we have*

$$B_{\overline{\Phi}}(g_\infty) = \frac{1}{g(M)}\overline{\det(J(g_\infty,i))^{-l}}\mu_2(g_\infty)^l e(-tr(S\cdot\overline{g_\infty(i)}) \sum_{\substack{1\le j\le h(-d)\\ 1\le k\le g(M)/t(d)}}\Lambda(t_j x_k)^{-1}\overline{a(S_{j,k})}$$

Remark. This is a mild generalization of [23, (1-26)]. We present a proof below. But first, we need some preliminary results.

For any f a function on $\mathbb{H}_2$ and $g_\infty\in G(\mathbb{R})^+$ define

$$(f|g_\infty)(Z) = f(g_\infty(Z))\mu_2(g_\infty)^l\det(J(g_\infty,i))^{-l}.$$

Let M_2^{Sym} denote the space of symmetric two by two matrices. We shall think of M_2^{Sym} as a subgroup of G via $x\mapsto u(x)$.

Also, for any continuous function f on $G(\mathbb{Q})\backslash G(\mathbb{A})$ define

$$C_f(g) = \int_{M_2^{Sym}(\mathbb{Q})\backslash M_2^{Sym}(\mathbb{A})} f(u(X)g)\psi(\mathrm{tr}(SX))^{-1}dX.$$

The following lemma is the content of [23, (1-19)]. However it is not proved there, so for convenience we include a proof here.

Lemma 8.3.6. *Let $g_\infty \in G(\mathbb{R})^+, g_f \in GL_2(\mathbb{A}_f)$. We consider g_f as an element of $G(\mathbb{A}_f)$ via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & det(g)\cdot(g^{-1})^T \end{pmatrix}$. Then*

$$C_{\overline{\Phi}}(g_\infty g_f) = \overline{\det(J(g_\infty, i))^{-l}}\mu_2(g_\infty)^l a(g_f, S)e(-tr(S\cdot \overline{g_\infty}(i)))$$

where $a(g_f, T)$ is the (T)'th Fourier coefficient of $\overline{F|g_\mathbb{R}}$, i.e.

$$\overline{F|g_\mathbb{R}(Z)} = \sum_T a(g_f, T)e(-tr\ T\overline{Z}),$$

and $g_\mathbb{R}$ is defined by the equation $g_f = g_\mathbb{Q} g_\mathbb{R} g_K$ with $g_\mathbb{Q} \in G(\mathbb{Q}), g_\mathbb{R} \in G(\mathbb{R})^+, g_K \in \prod_{p<\infty, p\nmid M} K_p \cdot \prod_{p|M} I_p$.

Proof. Put $U_p = \prod_{p<\infty,p\nmid M} K_p \cdot \prod_{p|M} I_p \subset G(\mathbb{A}_f)$. Define $\overline{\Phi}_f$ by $\overline{\Phi}_f(g) = \overline{\Phi}(gg_f)$. Then $\overline{\Phi}_f$ is left invariant by $G(\mathbb{Q})$ and right invariant by $g_\mathbb{Q} U_p g_\mathbb{Q}^{-1}$. From that it follows that $\overline{\Phi}_f(gg_\infty) = \overline{\Phi}_f(g_\infty)$ if $g \in g_\mathbb{Q}(\prod_{q<\infty} U(\mathbb{Z}_q))g_\mathbb{Q}^{-1}$. Also note that $C_{\overline{\Phi}}(g_\infty g_f) = C_{\overline{\Phi}_f}(g_\infty)$ and

$$\overline{\Phi}_f(g_\infty) = \mu_2(g_\infty)^l \overline{\det(J(g_\infty, i))^{-l}}(\overline{F|g_\mathbb{R})(g_\infty(i)}).$$

Finally, by approximation, we have

$$M_2^{Sym}(\mathbb{A}) = M_2^{Sym}(\mathbb{Q}) + \det(g_\mathbb{Q})^{-1} g_\mathbb{Q}\left(M_2^{Sym}(\mathbb{R})\prod_{q<\infty} M_2^{Sym}(\mathbb{Z}_q)\right) g_\mathbb{Q}^T.$$

Therefore

$$\begin{aligned} C_{\overline{\Phi}_f}(g_\infty) &= \int_{M_2^{Sym}(\mathbb{Q})\backslash M_2^{Sym}(\mathbb{A})} \overline{\Phi}_f(u(X)g_\infty)\psi(\mathrm{tr}(SX))^{-1}dX \\ &= \int_{\det(g_\mathbb{Q})^{-1}g_\mathbb{Q} M_2^{Sym}(\mathbb{Z})g_\mathbb{Q}^T\backslash M_2^{Sym}(\mathbb{R})} \overline{\Phi}_f(u(X)g_\infty)e(\mathrm{tr}(SX))dX \\ &= \mu_2(g_\infty)^l\overline{\det(J(g_\infty,i))^{-l}}\sum_T a(g_f,T)e(\text{-}\mathrm{tr}(T\cdot\overline{g_\infty(i)})) \\ &\cdot\left(\int_{M_2^{Sym}(\mathbb{Z})\backslash M_2^{Sym}(\mathbb{R})} e(tr(T+S)\cdot X)dX\right) \\ &= \overline{\det(J(g_\infty,i))^{-l}}\mu_2(g_\infty)^l a(g_f,S)e(-\mathrm{tr}(S\cdot\overline{g_\infty(i)})) \end{aligned}$$

□

Proof of Proposition 8.3.5. The case $M = 1$ is proved in [23]. So we assume $M > 1$. Note that

$$B_{\overline{\Phi}}(g) = \int_{Z(\mathbb{A})T(\mathbb{Q})\backslash T(\mathbb{A})} C_{\overline{\Phi}}(tg)\Lambda^{-1}(t)dt.$$

Hence, using (8.3.5) and the fact that $C_{\overline{\Phi}}$ is right invariant by $\Pi_{p<\infty,p\nmid M}T(\mathbb{Z}_p)\cdot\Pi_{p|M}\Gamma^0_{L,p}$ we have

(8.3.11)

$$B_{\overline{\Phi}}(g_\infty) = [SL_2(\mathbb{Z}) : \Gamma^0(M)]^{-1}\sum_{j,k}\Lambda^{-1}(t_jx_k)\int_{Z_T(\mathbb{R})\backslash T(\mathbb{R})} C_{\overline{\Phi}}(t_jx_kt_\infty g_\infty)dt_\infty.$$

Our Haar measure is normalized so that the compact set $Z_T(\mathbb{R})\backslash T(\mathbb{R})$ has volume 1. We henceforth write R^* instead of $Z_T(\mathbb{R})$ for simplicity. We have,

(8.3.12)

$$\begin{aligned}
&\int_{R^*\backslash T(\mathbb{R})} C_{\overline{\Phi}}(t_j x_k t_\infty g_\infty) dt_\infty \\
&= \int_{R^*\backslash T(\mathbb{R})} C_{\overline{\Phi}}(t_\infty g_\infty t_j x_k) dt_\infty \\
&= \int_{R^*\backslash T(\mathbb{R})} C_{\overline{\Phi}}(t_\infty g_\infty (\gamma_{j,k})_f) dt_\infty \\
&= \int_{R^*\backslash T(\mathbb{R})} \overline{\det(J(t_\infty g_\infty, i))^{-l}} \mu_2(t_\infty g_\infty)^l a((\gamma_{j,k})_f, S) e(-\mathrm{tr}(S \cdot t_\infty \overline{g_\infty}(i))) dt_\infty \\
&= \overline{\det(J(g_\infty, i))^{-l}} \mu_2(g_\infty)^l a((\gamma_{j,k})_f, S) \left(\int_{R^*\backslash T(\mathbb{R})} e(-\mathrm{tr}(S \cdot \overline{g_\infty}(i))) dt_\infty\right) \\
&= \overline{\det(J(g_\infty, i))^{-l}} \mu_2(g_\infty)^l a((\gamma_{j,k})_f, S) e(-\mathrm{tr}(S \cdot \overline{g_\infty(i)}))
\end{aligned}$$

Let us compute $a((\gamma_{j,k})_f, S)$. We have

$$\begin{aligned}
\overline{F|m_{j,k}(Z)} &= \sum_{T>0} \overline{a(T)} e(-\mathrm{tr}\ T \cdot (m_{j,k}(\overline{Z}))) \\
&= \sum_{T>0} \overline{a(T)} e(-\mathrm{tr} \det(m_{j,k}^{-1}) \cdot ((m_{j,k})^T T m_{j,k}) \cdot \overline{Z}).
\end{aligned}$$

So, the S'th Fourier coefficient corresponds to $T = \det(m_{j,k})\ (m_{j,k}^{-1})^T S m_{j,k}^{-1} = S_{j,k}$. Thus

$$a((\gamma_{j,k})_f, S) = \overline{a(S_{j,k})}. \tag{8.3.13}$$

Putting together (8.3.11),(8.3.12) and (8.3.13), we have the proof of the proposition.

□

8.4. **Description of the Eisenstein series.** This section describes the Eisenstein series on $\widetilde{G}(\mathbb{A})$. For each finite place v, recall that $\widetilde{K_v}$ is the maximal compact subgroup of $\widetilde{G}(\mathbb{Q}_v)$ and is defined by

$$\widetilde{K_v} = \widetilde{G}(\mathbb{Q}_v) \cap GL_4(\mathbb{Z}_{L,v}).$$

Let us now define

$$\widetilde{K_\infty} = \{g \in \widetilde{G}(\mathbb{R}) | \mu_2(g) = 1, g < iI_2 >= iI_2\}.$$

Equivalently

$$\widetilde{K_\infty} = U(2,2;\mathbb{R}) \cap U(4,\mathbb{R}).$$

We define

$$\rho_l(k_\infty) = \det(k_\infty)^{l/2} \det(J(k_\infty, i))^{-l}.$$

By [11, p. 5], any matrix k_∞ in $\widetilde{K_\infty}$ can be written in the form $k_\infty = \lambda \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ where $\lambda \in \mathbb{C}, |\lambda| = 1$, and $A+iB, A-iB \in U(2;\mathbb{R})$ with $\det(A+iB) = \overline{\det(A-iB)}$. Then,

$$\rho_l(k_\infty) = \det(A - iB)^{-l} \tag{8.4.1}$$

Note that if k_∞ has all real entries, i.e. $k_\infty \in \mathrm{Sp}(4,\mathbb{R}) \cap \mathrm{O}(4,\mathbb{R})$, then

$$\rho_l(k_\infty) = \det(J(k_\infty, i))^{-l}.$$

Extend Ψ to $GU(1,1;L)(\mathbb{A})$ by

$$\Psi(ag) = \Psi(g)$$

for $a \in L^\times(\mathbb{A}), g \in GL_2(\mathbb{A})$. Now define the compact open subgroup $K^{\widetilde{G}}$ of $\widetilde{G}(\mathbb{A}_f)$ by

$$K^{\widetilde{G}} = \prod_{p<\infty, p\nmid MN} \widetilde{K_p} \prod_{p|N, p\nmid M} \widetilde{U_r} \prod_{p|M} I'_p$$

Let us now define an element $\widetilde{k} \in \widetilde{G}(\mathbb{A})$ as follows. Let $\widetilde{k} = \prod_v \widetilde{k}_v$ where

$$\widetilde{k}_v = \begin{cases} \Theta_v & \text{if } v \text{ is a finite prime such that } v|M, v \nmid N, \\ 1 & \text{otherwise.} \end{cases} \tag{8.4.2}$$

Define

$$f_\Lambda(g,s) = \delta_P^{s+\frac{1}{2}}(m_1 m_2)\Lambda(\overline{m_1})^{-1}\Psi(m_2)\rho_l(k_\infty) \quad \text{if } g = m_1 m_2 n \widetilde{k} k \tag{8.4.3}$$

where $m_i \in M^{(i)}(\mathbb{A}) \quad (i = 1,2)$, $n \in N(\mathbb{A})$, $k = k_\infty k_0$ with $k_\infty \in \widetilde{K_\infty}$, $k_0 \in K^{\widetilde{G}}$ and $\widetilde{k} = \prod_{p|M} k_p$ is such that $k_p \in \{1, s_1\}$ for $p|\gcd(M,N)$ and $k_p \in \{1, \Theta\}$ for $p|M, p \nmid N$, and put

$$f_\Lambda(g,s) = 0$$

otherwise.

Finally, we define the Eisenstein series $E_{\Psi,\Lambda}(g,s)$ on $\widetilde{G}(\mathbb{A})$ by

$$E_{\Psi,\Lambda}(g,s) = \sum_{\gamma \in P(\mathbb{Q})\backslash \widetilde{G}(\mathbb{Q})} f_\Lambda(\gamma g, s). \tag{8.4.4}$$

8.5. **The global integral.** The global integral for our consideration is

$$Z(s) = \int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} E_{\Psi,\Lambda}(g,s)\overline{\Phi}(g)dg.$$

Then, by (2.2.3), Theorem 2.3.1, Theorem 5.3.1, Theorem 6.3.1 and Theorem 7.3.1 we have

(8.5.1)

$$Z(s) = \frac{Q_f Z_\infty(s)}{g(M/f)P_{MN}} \cdot \prod_{p|f} \frac{p^{-6s-3}}{1 - a_p w_p p^{-3s-3/2}} \cdot \frac{L(3s+\frac{1}{2}, \pi \times \sigma)}{\zeta_{MN}(6s+1)L(3s+1, \sigma \times \rho(\Lambda))}$$

where f denotes $\gcd(M,N)$ and

$$L(s, \pi \times \sigma) = \prod_{q<\infty} L(s, \pi_q \times \sigma_q)$$

$$L(s, \sigma \times \rho(\Lambda))) = \prod_{q<\infty, q\nmid M} L(s, \sigma_q \times \rho(\Lambda_q)),$$

$$\zeta_A(s) = \prod_{\substack{p\nmid A \\ p \text{ prime}}} (1 - p^{-s})^{-1},$$

$$P_A = \prod_{\substack{r|A \\ r \text{ prime}}} (r^2 + 1),$$

,

$$Q_A = \prod_{\substack{r|A \\ r \text{ prime}}} (1-r),$$

and

$$Z_\infty(s) = \int_{R(\mathbb{R})\backslash G(\mathbb{R})} W_{f_\Lambda}(\Theta g, s) B_{\overline{\Phi}}(g) dg \tag{8.5.2}$$

As for the explicit computation of Z_∞, Furusawa's calculation in [2], *mutatis mutandis*, works for us. The only real point of difference is the choice of S. Furusawa chooses

$$S = \begin{cases} \begin{pmatrix} \frac{d}{4} & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } d \equiv 0 \pmod 4, \\ \begin{pmatrix} \frac{1+d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

He computes $Z_\infty(s)$ for the case $d \equiv 0 \pmod 4$ and uses it to deduce the other case via a simple change of variables, using

$$\begin{pmatrix} \frac{1+d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}^T \begin{pmatrix} \frac{d}{4} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

In our case we have,

$$S = \begin{pmatrix} \frac{b^2+d}{4} & \frac{b}{2} \\ \frac{b}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b}{2} & 1 \end{pmatrix}^T \begin{pmatrix} \frac{d}{4} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{b}{2} & 1 \end{pmatrix}$$

and so a similar change of variables works.

Define $a(\Lambda) = a(F, \Lambda)$ by

$$a(\Lambda) = \begin{cases} \sum_{1 \le j \le h(-d)} \Lambda(t_j) a(S_j) & \text{if } M = 1 \\ \frac{1}{g(M)} \sum_{\substack{1 \le j \le h(-d) \\ 1 \le k \le g(M)/t(d)}} \Lambda(t_j x_k)^{-1} \overline{a(S_{j,k})} & \text{if } M > 1. \end{cases}$$

Then we have (cf. [2, p. 214])

$$Z_\infty(s) = \pi \overline{a(\Lambda)} (4\pi)^{-3s - \frac{3}{2}l + \frac{3}{2}} d^{-3s - \frac{l}{2}} \cdot \frac{\Gamma(3s + \frac{3}{2}l - \frac{3}{2})}{6s + l - 1}.$$

Henceforth we simply write $L(s, F \times g)$ for $L(s, \pi \times \sigma)$. We can summarize our computations in the following theorem.

Theorem 8.5.1 (The integral representation)**.** *Let F and $E_{\Psi,\Lambda}$ be as defined previously. Then*

$$\int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} E_{\Psi,\Lambda}(g, s) \overline{\Phi}(g) dg = C(s) \cdot L(3s + \frac{1}{2}, F \times g)$$

where $C(s) =$

$$\frac{A(f)\pi \overline{a(\Lambda)} (4\pi)^{-3s - \frac{3}{2}l + \frac{3}{2}} d^{-3s - \frac{l}{2}} \Gamma(3s + \frac{3}{2}l - \frac{3}{2})}{g(M/f) P_{MN}(6s + l - 1) \zeta_{MN}(6s + 1) L(3s + 1, \sigma \times \rho(\Lambda))} \prod_{p|f} \frac{p^{-6s-3}}{1 - a_p w_p p^{-3s-3/2}}$$

with $f = \gcd(M, N)$.

Remark. Note that

$$C(\frac{l}{6}-\frac{1}{2}) = \frac{\pi^{4-2l} \quad \overline{a(F,\Lambda)}}{\zeta(l-2)L(\frac{l-1}{2},\sigma\times\rho(\Lambda))} \times \text{ (an algebraic number).}$$

9. A classical reformulation and special value consequences

Let

$$\widetilde{G}^+(\mathbb{R}) = \{g \in \widetilde{G}(\mathbb{R}) : \mu_2(g) > 0\},$$
$$G^+(\mathbb{R}) = \{g \in G(\mathbb{R}) : \mu_2(g) > 0\}.$$

Also, define

$$\widetilde{\mathbb{H}}_2 = \{Z \in M_4(\mathbb{C}) | i(\overline{Z} - Z) \text{ is positive definite}\}.$$

Note that $\widetilde{G}^+(\mathbb{R})$ acts transitively on $\widetilde{\mathbb{H}}_2$. For $g \in \widetilde{G}^+(\mathbb{R})$, $z \in \widetilde{\mathbb{H}}_2$, define $J(g,z)$ in the usual manner.

For $Z = \begin{pmatrix} * & * \\ * & z_{22} \end{pmatrix} \in \widetilde{\mathbb{H}}_2$, we set $\widehat{Z} = \frac{i}{2}(\overline{Z}^T - Z)$ and $Z^* = z_{22}$.

Now, let us interpret the Eisenstein series of the last section as a function on $\widetilde{\mathbb{H}}_2$. Recall the definitions of the global section $f_\Lambda(g,s) \in \mathrm{Ind}_{P(\mathbb{A})}^{\widetilde{G}(\mathbb{A})}(\Pi \times \delta_P^s)$, and the corresponding Whittaker function $W_{f_\Lambda} = \prod_v W_{f_\Lambda, v}$.

Also for $z \in \mathbb{H}_2$, put

$$W'(z) = \det(g)^{-l/2} J(g,i)^l W_\Psi(g)$$

where W_Ψ is the Whittaker function associated to Ψ and $g \in GL_2^+(\mathbb{R})$ is any element such that $g(i) = z$. Note that this definition does not depend on g.

Lemma 9.0.2. *Let $g_\infty \in \widetilde{G}^+(\mathbb{R})$. Then*

$$W_{f_\Lambda,\infty}(g_\infty, s) = \det(g_\infty)^{l/2} \det(J(g_\infty,i))^{-l} \left(\frac{\det(\widehat{g_\infty(i)})}{\mathrm{Im}(g_\infty(i))^*} \right)^{3(s+\frac{1}{2})-\frac{l}{2}} W'((g_\infty(i)^*).$$

Thus the function

$$\det(g_\infty)^{-l/2} \det(J(g_\infty,i))^l W_{f_\Lambda,\infty}(g_\infty,s)$$

depends only on $g_\infty(i)$.

Proof. Let us write

$$g_\infty = m^{(1)}(a) m^{(2)}(b) n k$$

where we use the notation of Subsection 1.2 with $a \in \mathbb{R}^\times$, $b = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2^+(\mathbb{R})$, $n \in N(\mathbb{R})$ and $k \in \widetilde{K}_\infty$. Observe that $b(i) = (g_\infty(i))^*$. Then, (8.4.3) tells us that

(9.0.3)
$$W_{f_\Lambda}(g_\infty, s) = |a^2 \mu_2(b)|^{3(s+\frac{1}{2}} \det(k)^{l/2} \det(b)^{l/2} J(b,i)^{-l} \det(J(k,i))^{-l} W'((g_\infty(i)^*).$$

On the other hand, we can verify that

$$\det(\widehat{g_\infty(i)}) = \mu_2(b)^2 |\det(J(g_\infty,i))|^{-2}. \tag{9.0.4}$$

Also,

$$\det(J(g_\infty,i)) = a^{-1}\mu_2(b)(\gamma i + \delta) \det(J(k,i)) \tag{9.0.5}$$

and

$$\mathrm{Im}(g_\infty(i))^* = \mu_2(b)|\gamma i + \delta|^{-2}. \tag{9.0.6}$$

Putting the above equations together, and using the fact that $|\det(J(k,i))|^{-2} = 1$, we get the statement of the lemma. ☐

Corollary 9.0.3. *Let $s \in \mathbb{C}$ be fixed. Then the function*

$$\det(g_\infty)^{-l/2} \det(J(g_\infty, i))^l E_{\Psi,\Lambda}(g_\infty, s)$$

depends only on $g_\infty(i)$.

Proof. Put

$$r_\lambda = \begin{pmatrix} 1 & & & \\ & \lambda & & \\ & & \lambda & \\ & & & 1 \end{pmatrix}.$$

The corollary follows immediately from the above lemma and the definition

$$E_{\Psi,\Lambda}(g_\infty, s) = \sum_{\lambda \in \mathbb{Q}} \sum_{\gamma \in P(\mathbb{Q}) \backslash \widetilde{G}(\mathbb{Q})} W_{f_\Lambda,\infty}((r_\lambda)_\infty \gamma_\infty g_\infty, s) \left(\prod_{v<\infty} W_{f_\Lambda,v}((r_\lambda)_v, \gamma_v s) \right).$$

☐

Define the function $\mathcal{E}(Z,s)$ on $\widetilde{H}_2$ by

$$\mathcal{E}(Z,s) = \det(g_\infty)^{-l/2} \det(J(g_\infty, i))^l E_{\Psi,\Lambda}(g_\infty, \frac{s}{3} + \frac{l}{6} - \frac{1}{2}). \tag{9.0.7}$$

We know [2] that the series defining $\mathcal{E}(Z,s)$ converges absolutely and uniformly for $s > 3 - \frac{l}{2}$. From now on, assume $l > 6$. Then $\mathcal{E}(Z,0)$ is a holomorphic Eisenstein series on $\widetilde{H}_2$. By [6] we know that $\mathcal{E}(Z,0)$ has algebraic Fourier coefficients.

Now, we consider the restriction of $\mathcal{E}(Z,0)$ to $\mathbb{H}_2$. Clearly, the resulting function also has algebraic Fourier coefficients.

Henceforth we abuse notation by using $\mathcal{E}(Z,0)$ to mean its restriction to $\mathbb{H}_2$.

Proposition 9.0.4. *Suppose $l > 6$. Then $\mathcal{E}(Z,0)$ is a Siegel modular form of weight l for $B(M) \cap U_2(N)$.*

Proof. By the above comments, $\mathcal{E}(Z,0)$ is holomorphic as a function on $\mathbb{H}_2$. Let $\gamma \in B(M) \cap U_2(N)$. We consider γ as an element of $G(\mathbb{Q})$ embedded diagonally in $G(\mathbb{A})$. Write $\gamma = \gamma_\infty \gamma_f$ where γ_f denotes the finite part. It suffices to show that

$$\mathcal{E}(\gamma_\infty Z, 0) = \det(J(\gamma_\infty, Z))^l \mathcal{E}(Z,0)$$

for $Z \in \mathbb{H}_2$.

Let $g \in Sp(4,\mathbb{R})$ be such that $g(i) = Z$; thus $\gamma_\infty g(i) = \gamma_\infty Z$.

We have

$$\begin{aligned} \mathcal{E}(\gamma_\infty Z, 0) &= \det(\gamma_\infty g)^{-l/2} \det(J(\gamma_\infty g, i))^l E_{g,\Lambda}(\gamma_\infty g, \frac{l}{6} - \frac{1}{2}) \\ &= \det(g)^{-l/2} \det(J(\gamma_\infty, Z))^l \det(J(g,i))^l E_{g,\Lambda}(\gamma g (\gamma_f)^{-1}, \frac{l}{6} - \frac{1}{2}) \\ &= \det(J(\gamma_\infty, Z))^l (\det(g)^{-l/2} \det(J(g,i))^l E_{g,\Lambda}(g, \frac{l}{6} - \frac{1}{2})) \\ &= \det(J(\gamma_\infty, Z))^l \mathcal{E}(Z,0) \end{aligned}$$

☐

For any congruence subgroup Γ of $Sp(4,\mathbb{Z})$ let $V(\Gamma)$ denote the quantity $[Sp(4,\mathbb{Z}) : \Gamma]^{-1}$.

Suppose $f(Z)$ and $g(Z)$ are Siegel modular forms of weight l for some congruence subgroup. We define the Petersson inner product

$$\langle f, g\rangle = \frac{1}{2}V(\Gamma)\int_{\Gamma\backslash\mathbb{H}_2} f(Z)\overline{g(Z)}(\det(Y))^{l-3}dXdY$$

where $Z = X + iY$ and Γ is any congruence subgroup such that f, g are both Siegel modular forms for Γ. Note that this definition does not depend on the choice of Γ.

Also for brevity, we put $\Gamma_{M,N} = B(M)\cap U_2(N)$ and $V_{M,N} = V(\Gamma_{M,N})$.

Proposition 9.0.5. *Assume $l > 6$. Define the global integral $Z(s)$ as in (8.5). Then*

$$Z(\frac{l}{2}-\frac{1}{2}) = \langle\mathcal{E}(Z,0), F\rangle.$$

Proof. By definition, we have

$$Z(\frac{l}{2}-\frac{1}{2}) = \int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} E_{\Psi,\Lambda}(g,0)\overline{\Phi}(g)dg.$$

It suffices to prove that

(9.0.8)
$$\int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} E_{\Psi,\Lambda}(g,s)\overline{\Phi}(g)dg = \frac{V_{M,N}}{2}\int_{\Gamma_{M,N}\backslash\mathbb{H}_2}\mathcal{E}(Z,0)\overline{F(Z)}\det(Y)^{l-3}dXdY.$$

Recall the definition of the compact open subgroup $K^{\widetilde{G}}$ from Subsection 8.4. The integrand on the left side is right invariant under $K^G = (K^{\widetilde{G}}K_\infty)\cap G(\mathbb{A})$. Furthermore $\mathrm{vol}(K^G) = V_{M,N}$ and we have

$$Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/K^G = \Gamma_{M,N}\backslash\mathbb{H}_2.$$

Now (9.0.8) follows from the above comments and the observation that the $G(\mathbb{R})^+$-invariant measure on $\mathbb{H}_2$ and dg are related by $dg = \frac{1}{2}(\det(Y))^{-3}dXdY$. □

For $\sigma\in\mathrm{Aut}(\mathbb{C})$, and an arbitrary Siegel modular form Θ, denote by Θ^σ (resp Θ^-) the Siegel modular form obtained by applying σ (resp. complex conjugation) to all the Fourier coefficients of Θ.

Theorem 9.0.6. *Let F, g be as defined in Subsection 8.2 with $l > 6$. Then, for $\sigma\in Aut(\mathbb{C}/\overline{\mathbb{Q}})$, we have*

$$\left(\frac{L(\frac{l}{2}-1, F\times g)}{\pi^{5l-8}\langle F, F^-\rangle\langle g, g\rangle}\right)^\sigma = \frac{L(\frac{l}{2}-1, F\times g)}{\pi^{5l-8}\langle F^\sigma, (F^\sigma)^-\rangle\langle g, g\rangle}.$$

Proof. From the the theorem at the top of p.460 in [3] we have

$$\left(\frac{\langle\mathcal{E}(Z,0), F^-\rangle}{\langle F, F^-\rangle}\right)^\sigma = \frac{\langle\mathcal{E}(Z,0)^\sigma, (F^\sigma)^-\rangle}{\langle F^\sigma, (F^\sigma)^-\rangle}$$

Now, we know that $\mathcal{E}(Z,0)^\sigma = \mathcal{E}(Z,0)$. Also, since all the Hecke eigenvalues of F are totally real and algebraic, we have

$$L(F\times g) = L(F^\sigma\times g) = L(F^-\times g).$$

Therefore, from the above proposition and the remark at the end of Theorem 8.5.1, it follows that

(9.0.9)
$$\left(\frac{\pi^{4-2l}\overline{a(F^-,\Lambda)}L(\frac{l}{2}-1,F\times g)}{\zeta(l-2)L(\frac{l-1}{2},\sigma\times\rho(\Lambda))\langle F,F^-\rangle}\right)^{\sigma}=\frac{\pi^{4-2l}\overline{a((F^{\sigma})^-,\Lambda)}L(\frac{l}{2}-1,F\times g)}{\zeta(l-2)L(\frac{l-1}{2},\sigma\times\rho(\Lambda))\langle F^{\sigma},(F^{\sigma})^-\rangle}.$$

It is well-known that $\zeta(l-2)\pi^{2-l}\in\mathbb{Q}$. Also using the same argument as in the proof of [2, Theorem 4.8.3], we have

$$\frac{L(\frac{l-1}{2},\sigma\times\rho(\Lambda))}{\pi^{2l-2}\langle g,g\rangle}\in\overline{\mathbb{Q}}.$$

These facts, when substituted in (9.0.9) give the assertion of the theorem.

□

The above theorem implies the following corollary.

Corollary 9.0.7. *Let F, g be as defined in Subsection 8.2 with $l > 6$ and furthermore assume that F has totally real algebraic Fourier coefficients. Then*

$$\frac{L(\frac{l}{2}-1,F\times g)}{\pi^{5l-8}\langle F,F\rangle\langle g,g\rangle}\in\overline{\mathbb{Q}}.$$

Remark. Newforms for $GL(2)$, when normalized, automatically have algebraic Fourier coefficients. A similar statement is not known for Siegel newforms (among other things, we do not know multiplicity one for $GSp(4)$). However by [3] we do know the following: The space of Siegel cusp forms for a principal congruence subgroup has a *basis* of Hecke eigenforms with totally real algebraic Fourier coefficients.

10. Further questions

It is of interest to investigate the special values of $L(s, F\times g)$ more closely. In particular, we may ask the following questions.

(a) Does the expected reciprocity law hold for the special value $L(\frac{l}{2}-1,F\times g)$? In other words, can one extend Theorem 9.0.6 to the case where σ is any automorphism of $\mathbb{C}$?
(b) Do we have similar special value results for the other 'critical' values of $L(s,F\times g)$ as predicted by Deligne's conjectures?

The above questions reduce to ones about the Eisenstein series that appears in the statement of Theorem 8.5.1. Indeed, to answer the first question, it suffices to know the behavior of the Fourier coefficients of $\mathcal{E}(Z,0)$ under an automorphism of $\mathbb{C}$. For the second, we would like to know similar facts for $\mathcal{E}(Z,s)$ with s lying outside the range of absolute convergence of the Eisenstein series. It seems hard to extract these directly, as our Eisenstein series — being induced from an automorphic representation of $GL(2)$ sitting inside the Klingen parabolic — is rather complicated.

However, using a 'pullback formula', we can switch to a more standard Siegel-type Eisenstein series on a higher rank group. More precisely, we will derive, in a sequel to this paper [20], another integral representation for the L-function which involves an Eisenstein series on $GU(3,3)$. Incidentally, this second integral representation looks very similar to the Garrett–Piatetski-Shapiro–Rallis integral representation for the triple product L-function.

Let us describe this second integral representation in more detail.

Let $\widetilde{G}^{(3)} = GU(3,3;L), \widetilde{F} = GU(1,1;L)$. Let H_1 denote the subgroup of $G \times \widetilde{F}$ consisting of elements $h = (h_1, h_2)$ such that $h_1 \in G, h_2 \in \widetilde{F}$ and $\mu_2(h_1) = \mu_1(h_2)$. We fix a certain embedding $H_1 \hookrightarrow \widetilde{G}^{(3)}$. Let $P_{\widetilde{G}^{(3)}}$ be the Siegel parabolic of $\widetilde{G}^{(3)}$. Given a section $\Upsilon(s)$ of $\mathrm{Ind}_{P_{\widetilde{G}^{(3)}}}^{\widetilde{G}^{(3)}}(\Lambda \times |\,|^{3s})$ define the Eisenstein series $E_\Upsilon(h,s)$ on $\widetilde{G}^{(3)}(\mathbb{A})$ in the usual manner.

Now consider the global integral

$$Z(s) = \int_{Z_{\widetilde{G}^{(3)}}(\mathbb{A})H_1(\mathbb{Q})\backslash H_1(\mathbb{A})} \Lambda^{-1}(\det h_2)\overline{\Phi}(h_1)\Psi(h_2)E_\Upsilon(h_1,h_2,s)dh$$

where $h = (h_1, h_2)$. Using the pullback formula, we will prove in [20] that for a suitable choice of Υ,

$$Z(s) = L\left(3s + \frac{1}{2}, F \times g\right) \times (\text{normalizing factor}).$$

So, to answer the questions stated in the beginning of this section it suffices to study the (simpler) Eisenstein series $E_\Upsilon(h,s)$. Indeed, the action of $\mathrm{Aut}(\mathbb{C})$ on the Fourier coefficients is then known, enabling us to answer the first question. For the second there seem to be two possible strategies: the theory of nearly holomorphic functions due to Shimura [22], or a Siegel-Weil formula based attack explained by Harris in his papers [7, 8].

In [20], the approach sketched in this section will be fleshed out and the special value properties of the L-function investigated in more detail.

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