Superstatistics in hydrodynamic turbulence

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Abstract

Superstatistics is a 'statistics of a statistics' relevant for driven nonequilibrium systems with fluctuating intensive parameters. It contains Tsallis statistics as a special case. We show that probability density functions of velocity differences and accelerations measured in Eulerian and Lagrangian turbulence experiments are well reproduced by simple superstatistics models. We compare fits obtained for log-normal superstatistics and χ^2-superstatistics (= Tsallis statistics).

1 Introduction

There is currently considerable interest in more general versions of statistical mechanics, known under the name nonextensive statistical mechanics [1, 2, 3]. In the mean time, it has become clear that Tsallis' original approach [1] can generalized in various ways, and that these techniques are often relevant for the effective description of nonequilibrium systems with strong fluctuations of an intensive parameter, where ordinary statistical mechanics has little to say [4, 5, 6]. A particular class of more general statistics relevant for nonequilibrium systems, containing Tsallis statistics as a special case, has been termed 'superstatistics' [6, 7, 8]. A superstatistics arises out of the superposition of two statistics, namely one described by ordinary Boltzmann factors $e^{-\beta E}$ and another one given by the probability distribution of β. This means the inverse temperature parameter β is assumed not to be constant but to be fluctuating on a relatively large time scale or spatial scale. Naturally, this kind of approach is physically relevant for driven nonequilibrium systems with fluctuations, rather than for equilibrium systems.

Depending on the probability distribution of β, there are infinitely many superstatistics. It has been shown that Tsallis statistics is a particular superstatistics obtained under the assumption that β is χ^2-distributed [5]. Various other examples of superstatistics have been studied [6], among them superstatistics of log-normal type. A main result of [6] was that for small E all superstatistics behave in a universal way, i.e. they generate probability distributions close to Tsallis distributions. But for large E the various superstatistics can have quite different properties.

In this paper we work out the application of this very new concept of statistical mechanics to fully developed turbulence. By comparison with various data from Eulerian and Lagrangian turbulence experiments, as well as data from direct numerical simulations (DNS) of the Navier Stokes equation, we will provide evidence that superstatistics of log-normal type quite well describes measured probability densities in hydrodynamic turbulence. This superstatistics can be dynamically realized by considering a class of stochastic differential equations previously introduced in [5], but now with a log-normal rather than χ^2-distribution of the damping parameter. In general, if the velocity difference is not too large, most superstatistics models yield probability densities that are similar to Tsallis statistics. Significant differences only occur for the tails of the distribution, i.e. for very rare events. For the extreme tails, superstatistics based on log-normal distributions seems to provide better fits than superstatistics based on χ^2-distributions (i.e. ordinary Tsallis statistics). On the other hand, if the velocity difference is not too large (say less than about 30 standard deviations), then Tsallis statistics is quite a good approximation, with the advantage that an explicit formula for the densities can be given. For 'early' work emphasizing the relevance of Tsallis statistics in 3d-turbulence, see e.g. [9, 10, 11].

We will analyse data sets from three different experiments/simulations. The experimental measurements were done by Swinney et al. (Eulerian turbulence) and Bodenschatz et al. (Lagrangian turbulence). The direct numerical simulation (DNS) data were obtained by Gotoh et al. We are very grateful to all three groups for providing us with their data.

2 Superstatistics and its dynamical realizations

2.1 The basic concept

Let us give a short introduction to the 'superstatistics' concept [6]. The idea is actually applicable to many systems, not only to turbulent systems. Consider a driven nonequilibrium systems with spatio-temporal fluctuations of an intensive parameter β. This can e.g. be the inverse temperature, or a chemical potential, or a function of the fluctuating energy dissipation in the flow (for the turbulence application). Locally, i.e. in cells where β is approximately constant, the system is described by ordinary statistical mechanics, i.e. ordinary Boltzmann factors $e^{-\beta E}$, where E is an effective energy in each cell. To describe the system in the long-term run, one has to do a spatio-temporal average over the fluctuating β. One obtains a superposition of two statistics (that of β and that of $e^{-\beta E}$), hence the name 'superstatistics'. One may define an effective Boltzmann factor $B(E)$ given by

$$B(E) = \int_0^\infty f(\beta) e^{-\beta E}, \tag{1}$$

where $f(\beta)$ is the probability distribution of β. For type-A superstatistics, one normalizes this effective Boltzmann factor, obtaining the stationary probability distribution

$$p(E) = \frac{1}{Z} B(E), \tag{2}$$

where

$$Z = \int_0^\infty B(E)dE. \tag{3}$$

For type-B superstatistics, one includes the β-dependent normalization constant into the averaging process, obtaining

$$p(E) = \int_0^\infty f(\beta)\frac{1}{Z(\beta)}e^{-\beta E}d\beta, \tag{4}$$

where $Z(\beta)$ is the normalization constant of $e^{-\beta E}$ for a given β. Both approaches can be easily mapped into each other, by defining a new probability density $\tilde{f}(\beta) \sim f(\beta)/Z(\beta)$. It is obvious that Type-B superstatistics with f is equivalent to type-A superstatistics with $\tilde{f}$.

A simple dynamical realization of a superstatistics can be constructed by considering stochastic differential equations with spatio-temporally fluctuating parameters [5]. Consider the Langevin equation

$$\dot{u} = \gamma F(u) + \sigma L(t), \tag{5}$$

where $L(t)$ is Gaussian white noise, $\gamma > 0$ is a friction constant, σ describes the strength of the noise, and $F(u) = -\frac{\partial}{\partial u}V(u)$ is a drift force. If γ and σ are constant then the stationary probability density of u is proportional to $e^{-\beta V(u)}$, where $\beta := \frac{\gamma}{\sigma^2}$ can be identified with the inverse temperature of ordinary statistical mechanics. Most generally, however, we may let the parameters γ and σ fluctuate so that $\beta = \frac{\gamma}{\sigma^2}$ has probability density $f(\beta)$. These fluctuations are assumed to be on a long time scale so that the system can temporarily reach local equilibrium. In this case one obtains for the conditional probability $p(u|\beta)$ (i.e. the probability of u given some value of β)

$$p(u|\beta) = \frac{1}{Z(\beta)}\exp\{-\beta V(u)\}, \tag{6}$$

for the joint probability $p(u, \beta)$ (i.e. the probability to observe both a certain value of u and a certain value of β)

$$p(u, \beta) = p(u|\beta)f(\beta) \tag{7}$$

and for the marginal probability $p(u)$ (i.e. the probability to observe a certain value of u no matter what β is)

$$p(u) = \int_0^\infty p(u|\beta)f(\beta)d\beta. \tag{8}$$

This marginal distribution is the generalized canonical distribution of the superstatistics considered. The above formulation corresponds to type-B superstatistics.

2.2 Application to turbulent systems

In the turbulence application, the mathematics is the same as outlined above, just the physical meaning of the variables u, β, E etc. is slightly different from that of an ordinary Brownian particle. First of all, u stands for a local velocity *difference* in the turbulent flow. On a very small time scale, this velocity difference is essentially the acceleration. It is really the velocity difference, not the velocity itself that we want to understand. Velocity differences in turbulence have been the subject of intensive investigations since the early work of Kolmogorov. The velocity itself is known to be approximately Gaussian, so we don't need any sophisticated model to understand this.

The basic idea is that turbulent velocity differences locally relax with a certain damping constant γ and are at the same time driven by rapidly fluctuating chaotic force differences. As a local momentum balance, we thus end up with eq. (5), where we model the chaotic force differences by Gaussian white noise. As has been shown in [12, 13], this approximation by Gaussian white noise can be made rigorous if the chaotic force differences act on a relatively small time scale as compared to γ^{-1} and if they have strong mixing properties.

Next, one knows that in turbulent flows the energy dissipation ϵ fluctuates in space and time. In our simple model, the dissipation process is described by the damping constant γ. It is thus most naturally to assume that the parameter β defined as $\beta := \gamma/\sigma^2$ is a simple function of the fluctuating energy dissipation in the flow. For a while, there is local relaxation (energy dissipation) with a certain value of $\beta = \gamma/\sigma^2$, then this parameter changes to a new value, and so on.

So unlike an ordinary Brownian particle, in the turbulence application u is not velocity but velocity difference, moreover β is not inverse temperature but a function of the fluctuating energy dissipation in the flow. Finally, the correct interpretation of E is that of an effective potential generating the relaxation dynamics of u, so for example $E = V(u) = \frac{1}{2}u^2$ generates a linear relaxation dynamics, whereas other functions $V(u)$ generate more complicated relaxation processes.

All one has to decide now is what the probability density of the parameter

β should be. It is known since the early papers by Kolmogorov in 1962 that it is reasonable to assume that the probability density of energy dissipation is approximately log-normal in a turbulent flow. Hence, if β is a simple power-law function of ϵ, this implies a lognormally distributed β. We thus end up in a most natural way with log-normal superstatistics. If β is a more complicated function of ϵ, we end up with other superstatistics.

The aim of our simple superstatistics models is neither to solve the turbulence problem nor to fully reproduce the spatio-temporal dynamics of the Navier-Stokes equation, but to have a very simple model that grasps some of the most important statistical properties of turbulence and at the same time is analytically tractable.

2.3 χ^2-superstatistics

In [5] a χ^2-distribution was chosen for $f(\beta)$,

$$f(\beta) = \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left\{\frac{n}{2\beta_0}\right\}^{\frac{n}{2}} \beta^{\frac{n}{2}-1} \exp\left\{-\frac{n\beta}{2\beta_0}\right\} \tag{9}$$

Here $\beta_0 = \int_0^\infty \beta f(\beta) d\beta$ is the average value of the fluctuating β, and n is a parameter of the χ^2-distribution. For $F(u) = -u$, i.e. linear damping forces described by $V(u) = \frac{1}{2}u^2$, the integral (8) is easily evaluated, and one obtains the result that the marginal distribution $p(u)$ is given by a Tsallis distribution

$$p(u) \sim \frac{1}{\left(1 + \frac{1}{2}\tilde{\beta}(q-1)u^2\right)^{\frac{1}{q-1}}}, \tag{10}$$

where the relation between the Tsallis parameters q, $\tilde{\beta}$ and the parameters n, β_0 of the χ^2-distribution is

$$q = 1 + \frac{2}{n+1} \tag{11}$$

$$\tilde{\beta} = \frac{2}{3-q}\beta_0. \tag{12}$$

The distribution has variance 1 for the choice $\tilde{\beta} = 2/(5-3q)$.

In turbulent flows, the assumption of a simple linear damping force may not be justified. More complicated nonlinear drift forces may effectively act.

If these forces are effectively described by power-law potentials of the form $V(u) \sim |u|^{2\alpha}$ one obtains for the marginal density $p(u)$ Tsallis distributions of the form

$$p(u) = \frac{1}{Z_q} \frac{1}{(1 + (q-1)\tilde{\beta}|u|^{2\alpha})^{\frac{1}{q-1}}}. \tag{13}$$

Formulas of this type were shown to very well fit densities of velocity differences u measured in a Taylor-Couette experiment [14]. Empirically one observes that the relation $\alpha = 2 - q$ is satisfied by the experimentally measured densities in this experiment. Using this relation, only one fitting parameter q remains, which is a function of the scale r on which the velocity differences are measured and of the Reynolds number. Excellent fits were obtained for all spatial scales and all accessible Reynolds numbers. The slight asymmetry of the measured distributions can be understood as well [11, 12, 14].

2.4 Log-normal superstatistics

Let us now proceed to log-normally distributed β. The log-normal distribution is given by

$$f(\beta) = \frac{1}{\beta s \sqrt{2\pi}} \exp\left\{ \frac{-(\log \frac{\beta}{\mu})^2}{2s^2} \right\}, \tag{14}$$

where μ and s are parameters. The average β_0 of the above log-normal distribution is given by $\beta_0 = \mu\sqrt{w}$ and the variance by $\sigma^2 = \mu^2 w(w-1)$, where $w := e^{s^2}$. Let us for the moment restrict ourselves to linear forces $F(u) = -u$. The integral given by (8)

$$p(u) = \frac{1}{2\pi s} \int_0^\infty d\beta \; \beta^{-1/2} \exp\left\{ \frac{-(\log \frac{\beta}{\mu})^2}{2s^2} \right\} e^{-\frac{1}{2}\beta u^2} \tag{15}$$

is the theoretical prediction for the stationary distribution of velocity differences in the turbulent flow if log-normal superstatistics is the correct model. The integral cannot be evaluated in closed form, but the equation is easily numerically integrated, and can be compared with experimentally measured densities $p(u)$. The distribution $p(u)$ has variance 1 for the choice $\mu = \sqrt{w}$, hence only one parameter s^2 remains if one compares with experimental data sets that have variance 1.

The moments for the log-normal superstatistics distribution (15) can be easily evaluated. All moments exist. The moments of a Gaussian distribution of variance β^{-1} are given by

$$\langle u^m \rangle_G = \frac{1}{\beta^{m/2}}(m-1)!! \tag{16}$$

(m even). Moreover, the moments of the lognormal distribution are given by

$$\langle \beta^m \rangle_{LN} = \mu^m w^{\frac{1}{2}m^2}. \tag{17}$$

Combining eq.(16) and (17) one obtains the moments of the superstatistics distribution $p(u)$ as

$$\begin{aligned} \langle u^m \rangle &= \langle \langle u^m \rangle_G \rangle_{LN} && (18) \\ &= (m-1)!! \langle \beta^{-m/2} \rangle_{LN} && (19) \\ &= (m-1)!! \mu^{-\frac{m}{2}} w^{\frac{1}{8}m^2} && (20) \end{aligned}$$

The variance is given by

$$\langle u^2 \rangle = \mu^{-1}\sqrt{w}. \tag{21}$$

All hyperflatness factors F_m are independendent of μ and given by

$$F_m := \frac{\langle u^{2m} \rangle}{\langle u^2 \rangle^m} = (2m-1)!! w^{\frac{1}{2}(m-1)}. \tag{22}$$

In particular, the flatness F_2 is given by

$$F_2 := \frac{\langle u^4 \rangle}{\langle u^2 \rangle} = 3w = 3e^{s^2}. \tag{23}$$

Measuring the flatness F_2 of some experimental data thus provides a very simple method to determine the fitting parameter s^2 of lognormal superstatistics.

In some recent work [16, 17], log-normal superstatistics and the generalized Langevin dynamics (5) is related to a generalized Sawford model for Lagrangian accelerations [18, 19]. This yields a power-law relation between ϵ and β. The relevance of distributions of similar form as in eq. (15) has also been emphasized in early work of Castaing et al. [20].

2.5 Other superstatistics

In principle, all kinds of distributions $f(\beta)$ can be considered, leading to different superstatistics models. Which distribution $f(\beta)$ is the most suitable one, depends on the physical problem under consideration. As mentioned above, for turbulent flows there are some arguments that $f(\beta)$ should be approximately log-normal. The log-normal distribution is probably still an approximation, it is not the last word, so presumably there are again some deviations from this and the ultimate superstatistics model that is the most relevant one to describe high-Reynolds number 3-dimensional turbulence is simply not known yet. Nevertheless, for any superstatistics one can define generalized entropies and (at least in principle) proceed to a generalized statistical mechanics description, following the ideas of [7]. A turbulent flow, by construction, is then a complex system of generalized statistical mechanics that maximizes the above generalized entropies subject to suitable constraints.

An interesting point is that all superstatistics reduce to Tsallis statistics for small effective energies E: For small E they all have the same quadratic first-order correction to the ordinary Boltzmann factor. This can be easily seen as follows. For any distribution $f(\beta)$ with average $\beta_0 := \langle\beta\rangle$ and variance $\sigma^2 := \langle\beta^2\rangle - \beta_0^2$ we can write

$$
\begin{aligned}
B &= \langle e^{-\beta E}\rangle \qquad (24)\\
&= e^{-\beta_0 E} e^{+\beta_0 E}\langle e^{-\beta E}\rangle \\
&= e^{-\beta_0 E}\langle e^{-(\beta-\beta_0)E}\rangle \\
&= e^{-\beta_0 E}\left(1+\frac{1}{2}\sigma^2 E^2 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m!}\langle(\beta-\beta_0)^m\rangle E^m\right). \qquad (25)
\end{aligned}
$$

Here the coefficients of the powers E^m are the m-th moments of the distribution $f(\beta)$ about the mean, which can be expressed in terms of the ordinary moments as

$$\langle(\beta-\beta_0)^m\rangle = \sum_{j=0}^{m}\binom{m}{j}\langle\beta^j\rangle(-\beta_0)^{m-j}. \qquad (26)$$

We see that for small E all superstatistics have a quadratic correction term to the ordinary Boltzmann factor, and the coefficient is the same as for Tsallis statistics ($= \chi^2$-superstatistics) if the distribution $f(\beta)$ is chosen with the same variance σ^2. In practice, one observes this 'universality' even for moderately large E: Many superstatistics are observed to yield pretty similar results $p(E)$ for moderately large E (see next section). Usually one observes significant differences only for very large values of E.

3 Comparison with experiments

3.1 Swinney's data on Taylor-Couette flow

Figs. 1 and 2 shows an experimentally measured $p(u)$ of velocity differences u at scale $r = 92.5\eta$ in a Taylor-Couette flow [14]. η denotes the Kolmogorov length scale. The data have been rescaled to variance 1. The Taylor scale Reynolds number is $R_\lambda = 262$. Apparently, there is excellent agreement between the measured density and log-normal superstatistics as given by eq. (15). The fitting parameter for this example is $s^2 = 0.28$. Note that s^2 is the only fitting parameter. The scale- and Reynolds number dependence of s^2 can be easily extracted from the measured flatness of the distributions, using eq. (23).

Fig. 3 shows that essentially the same curve as for log-normal superstatistics can be also obtained if one uses Tsallis statistics, i.e. eq. (13) with $q = 1.11$ and $\alpha = 2 - q$. Indeed, the two theoretical curves can hardly be distinguished in the experimentally relevant region of $|u| < 8$. Significant differences only arise for much larger $|u|$. So both types of superstatistics are compatible with the experimental data.

One theoretical advantage of log-normal superstatistics is that it does not require a nonlinear force $F(u)$, i.e. an α different from 1, to fit the data perfectly. A linear forcing is completely sufficient in that case. The only fitting parameter that we use for the log-normal superstatistics is s^2, since the parameter μ is fixed as $\mu = e^{\frac{1}{2}s^2}$ to give variance 1.

Swinney et al. have also measured the probability distribution of the shear stress S at the outer and inner cylinder in their Taylor-Couette experiment [15]. This distribution is well approximated by a log-normal distribution, at least for large values of S (see Fig. 4). The square of the shear stress is essentially the energy dissipation ϵ in the flow, and if

$$\beta = C \cdot \epsilon^\kappa \tag{27}$$

is some simple power-law function of ϵ then the measurements of the nearly log-normal shear stress distribution indicate that the superstatistics parameter β should be approximately log-normally distributed as well. There are indeed some theoretical arguments that suggest a power-law relation of type (27), e.g. with $\kappa = -3/2$, see [17] for details. Hence Swinney's measurements are an indirect experimental hint towards the physical relevance of log-normal superstatistics.

3.2 Bodenschatz's data on Lagrangian accelerations

Accelerations a of Lagrangian test particles in turbulent flows are in practice measured as velocity differences u on a small time scale τ of the order τ_η, the Kolmogorov time. Hence $a \approx u/\tau$. Fig. 5 shows the most recent measurements of histograms of accelerations of Lagrangian test particles as obtained in the experiment of Bodenschatz et al. [21, 22, 23]. The Reynolds number is $R_\lambda = 690$. The measured distributions are reasonably well approximated by Tsallis distributions of type (13) for moderately large accelerations (see e.g. [24] for a comparison with $\alpha = 1$ and Fig. 5 for a comparison with $\alpha = 0.5$). But for extremely large accelerations the data seem to systematically fall below curves corresponding to Tsallis statistics, at least if the exponent α of the potential V is kept in the physically reasonable range $\frac{1}{2} \leq \alpha \leq 1$. As shown in Fig. 5 as well, log-normal superstatistics provides a better fit of the tails, with $s^2 = 3.0$ and using just a linear damping force, i.e. $\alpha = 1$. Since Bodenschatz's data reach rather large accelerations a (in units of the standard deviation), the measured tails of the distributions allow for a sensitive distinction between various superstatistics. The main difference between χ^2-superstatistics and log-normal superstatistics is the fact that $p(a)$ decays with a power law for the former ones, whereas it decays with a more complicated logarithmic law for the latter ones. For alternative fitting approaches, see [25].

One remark is at order. The acceleration is actually experimentally determined as a parabolic fit of the measured position of the test particle on a finite time scale τ of the order of τ_η , or as a velocity difference on the same time scale. While in the early paper of the Bodenschatz group [21] no dependence of the data on τ was mentioned, in the later paper [22] a significant dependence of the flatness of the distributions on τ was described (Fig. 28 in [22]). The flatness of the distribution is significantly linked to the tails, larger flatness certainly means tails that lie higher. So the shape of all measured distributions actually depends on the seemingly arbitrary parameter τ. What the asymptotics is for $\tau \to 0$ depends on extrapolation assumptions. Even the existence of this limit is not clear at all. In addition, the tails will presumably still change shape with increasing Reynolds number. We are theoretically interested in the infinite Reynolds number case, which could still be very different from the finite-Reynolds number case. The infinite Reynolds number case could still be correctly described by Tsallis statistics. And finally, does a finite-size test particle in the experimental flow really follow completely the extremely strong forces in the flow, which are supposed to accelerate it to accelerations of up to 2000 g? No astronaut would survive this! So it may well be that the measured extreme tails of $p(a)$ contain some systematic negative corrections, simply because the particle cannot follow those extreme forces. Summa summarum, one should be very cautious when drawing over-ambitious conclusions that are solely based on fits of extreme tail data. The tails describe acceleration events that are a million times more unlikely than events near the maximum of the distribution.

3.3 Gotoh's DNS data

Fig. 6 shows Gotoh's results on the pressure distribution as obtained by direct numerical simulation of the Navier-Stokes equation at $R_\lambda = 380$ [26]. A direct numerical simulation is also a kind of experiment, just that it is done on a computer. One usually assumes that in reasonably good approximation the pressure statistics coincides with the acceleration statistics of a Lagrangian test particle. Gotoh's histograms reach accelerations up to 150 (in units of the standard deviation), a much larger statistics than can be presently reached in Bodenschatz's experiment. Hence the tails of these distributions can very sensitively be used to distinguish various superstatistics models.

Fig. 6 shows that log-normal superstatistics with $s^2 = 3.0$ and linear forcing again yields a good fit of the tails, keeping in mind that one compares data that vary over 12 orders of magnitude.

Near the maximum of the distributions, the fit quality of log-normal superstatistics is not very good: $p(0)$ is too big as compared to the DNS data. But this defect can be easily cured by introducing an upper cutoff in β. That is to say, in eq. (15) we only integrate up to a certain β_{max} and re-normalize afterwards. Log-normal superstatistics with an upper cutoff of $\beta_{max} \approx 32$ yields quite a perfect fit in the vicinity of the maximum (Fig. 7). The tails are not influenced by this cutoff. The above truncation may effectively represent finite size or finite Reynolds number effects, which are certainly present in any numerical simulation of the Navier-Stokes equation.

As Fig. 8 shows, Tsallis statistics with $q = 1.476$ and $\alpha = 0.832$ also yields a very good fit of the data in the vicinity of the maximum (the relation between q and α is $q = 1 + 2\alpha/(3\alpha + 1)$, the theoretical prediction of the model considered in [5] with $n = 3$ spatial dimensions). But for very large accelerations Tsallis statistics implies power-law tails, which are not supported by the finite-Reynolds number DNS data.

Of course the following general question arises: How much can we believe in the extreme tails of a DNS simulation? It should be clear that every DNS is a brute force finite lattice size approximation of the Navier-Stokes equation. Naturally there are finite-lattice size effects, also finite lattice constant effects, and moreover finite Reynolds number effects, which may heavily influence the extreme tails. Moreover, do the extreme events of 150 standard deviations, corresponding to accelerations of almost 10000 g, really describe plausible physics? Can a true physical test particle really follow such a force? Bodenschatz's experiment, tracking single test particles, leads to $p(a) \sim 10^{-8}$ for the most rare acceleration events. The detector for these measurements was running for about a month to collect the data. Gotoh's DNS data reach $p(a) \sim 10^{-12}$ for the rarest acceleration events. This statistics is larger by a factor 10^4. Hence Bodenschatz, in a laboratory experiment similar to the one he did so far, would need to wait 10^4 months ≈ 1000 years to observe one of the extreme acceleration events described by Gotoh's numerical simulation. I guess most physicists are not willing to wait that long.

4 Conclusion

By analyzing three different data sets obtained by Swinney, Bodenschatz, Gotoh, respectively, we have shown that measured and simulated densities in Eulerian and Lagrangian turbulence experiments are well described by simple superstatistics models. Log-normal superstatistics differs from χ^2-superstatistics, i.e. ordinary Tsallis statistics, but for moderately large velocity differences log-normal superstatistics can be quite close to Tsallis statistics, as shown e.g. in Fig. 3. The fact that this is so is not surprising but simply a consequence of the 'universality' property discussed in section 2.5. For small effective energies E (i.e. small u or a in the turbulence application) *any* superstatistics is close to Tsallis statistics. In practice we see that this is often also the case for moderately large velocity differences and accelerations. Significant differences only arise for very large velocity differences (and large accelerations), where Tsallis statistics predicts a power law decay of probability density functions, whereas log-normal superstatistics yields tails that decay in a more complicated way. It is indeed the tails contain the information on the most appropriate superstatistics for turbulent flows. A precise estimate of the error bars of the tails of experimentally measured or simulated distributions is clearly needed, taking into account not only statistical errors but all systematic errors as well. Moreover, one would wish for precise data on how the shape of the tails depends on the time scale on which the accelerations are measured, and how the tails change with Reynolds number. Finally, it would be interesting to have precise data on correlation functions of accelerations, since these yield more information than the densities alone.

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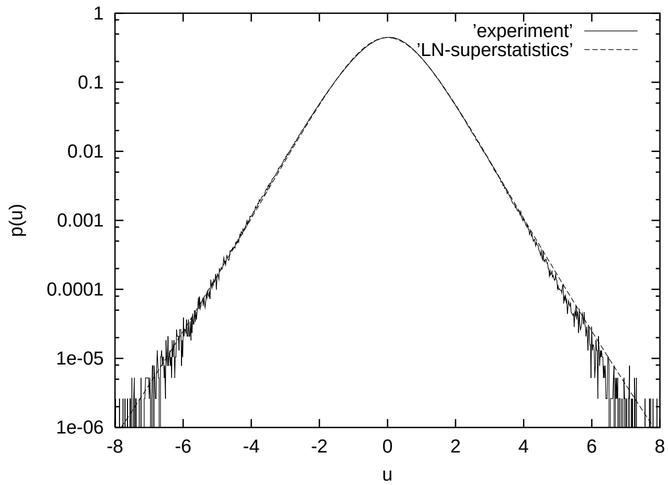

Fig. 1 Histogram of velocity differences u as measured in Swinney's experiment and the log-normal superstatistics prediction eq. (15) with $s^2 = 0.28$.

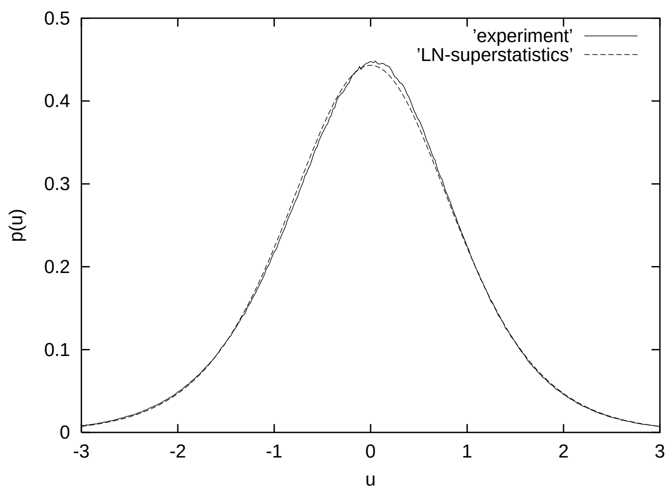

Fig. 2 Same as Fig. 1, but a linear scale is chosen. This emphasizes the vicinity of the maximum, rather than the tails.

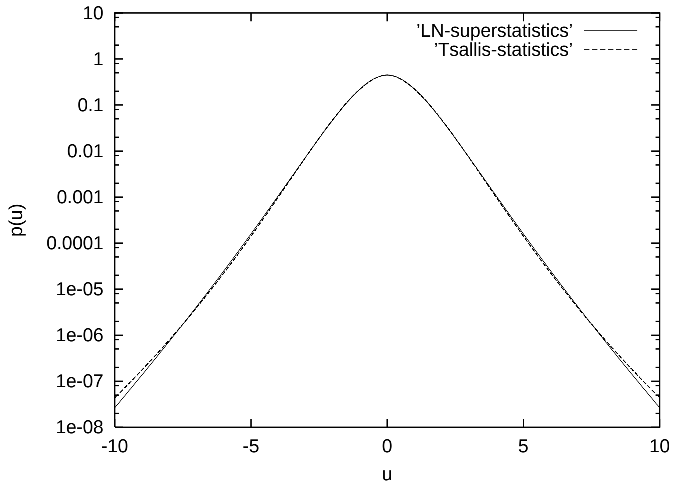

Fig. 3 Comparison between log-normal superstatistics as given by eq. (15) with $s^2 = 0.28$ and Tsallis statistics as given by eq. (13) with $q = 1.11$ and $\alpha = 2 - q$. For the range of values accessible in the experiment, $|u| < 8$, there is no visible difference between the two curves.

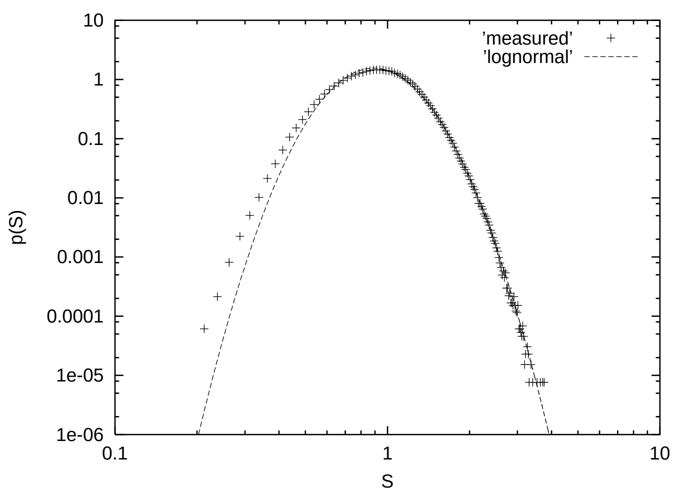

Fig. 4 Swinney's measurements of the shear stress distribution at the outer cylinder of the Taylor-Couette experiment, and comparison with a log-normal distribution.

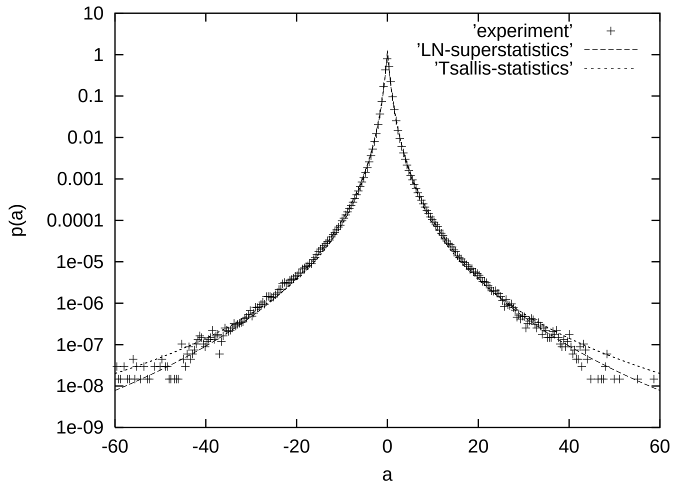

Fig. 5 Acceleration distribution as measured by Bodenschatz et al. and comparison with the log-normal superstatistics distribution (15) with $s^2 = 3.0$. Also shown is a Tsallis distribution (13) with $q = 1.2$ and $\alpha = 0.5$.

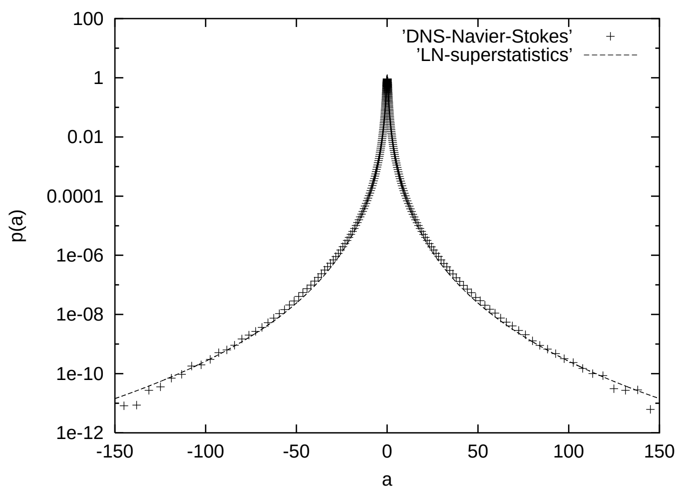

Fig. 6 Pressure statistics as obtained by Gotoh et al. in a direct numerical simulation of the Navier-Stokes equation, and comparison with log-normal superstatistics with $s^2 = 3.0$.

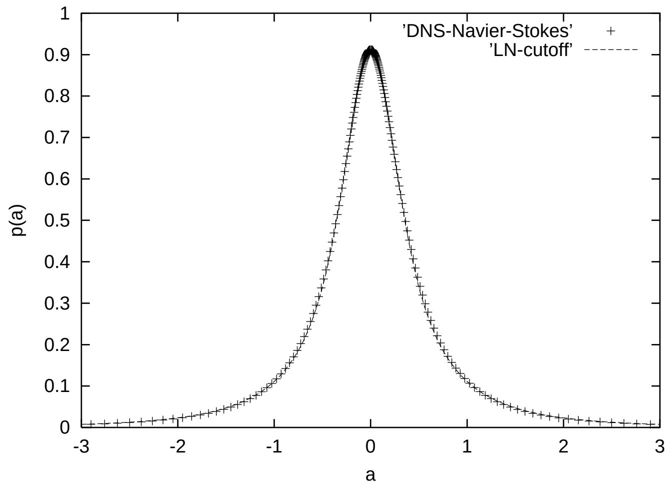

Fig. 7 Same data as in Fig. 6, but a linear scale is chosen to emphasize the vicinity of the maximum. The fitted line (hardly visible behind the data points) corresponds to log-normal superstatistics with $s^2 = 3.0$ and an upper cutoff $\beta_{max} = 32$.

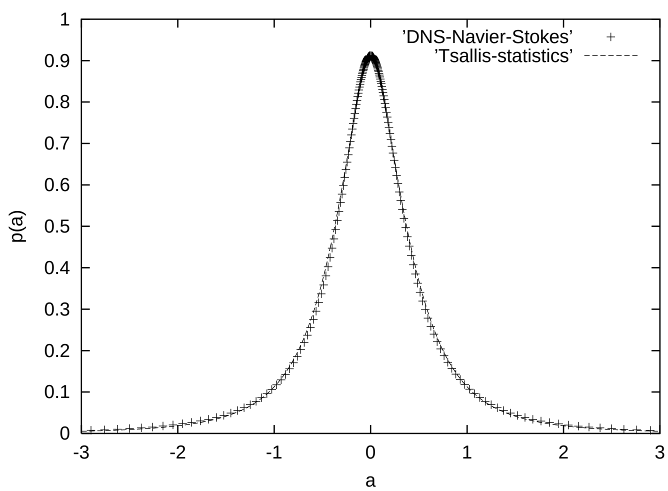

Fig. 8 Same data as in Fig. 7. The fitted line corresponds to Tsallis statistics with $q = 1.476$ and $\alpha = 0.832$. Only α is fitted–the value of q follows from formula (17) in [5].