Higher-order correlations of Tchebyscheff maps

Angela Hilgers[1]*and Christian Beck[2]
[1]*Institut für Theoretische Physik, RWTH Aachen*
Templergraben 55, 52056 Aachen,
Germany
[2]*School of Mathematical Sciences*
Queen Mary and Westfield College
University of London
Mile End Road, London E1 4NS
England

November 17, 1999

Abstract

We determine the higher-order correlation functions of iterates of Tchebyscheff polynomials of arbitrary order N. The non-vanishing correlations can be described by suitable sets of graphs. These graphs allow for a perturbative evaluation of invariant densities for Langevin-like systems that are driven by deterministic chaotic forces generated by Tchebyscheff maps rather than Gaussian white noise.

* *Present address: Center for Turbulence Research, Bldg. 500, Stanford University, Stanford, CA 94305-3030, USA*

1 Introduction

Regarding the initial values as random variables, any deterministic chaotic dynamical system generates a stochastic process. Generically, this process is non-Gaussian, and hence for a complete characterization the set of all higher-order correlation functions has to be evaluated. However, so far only very few papers or books have dealt with the set of n-point functions for chaotic dynamical systems [1]—[4].

In this paper we will carry out the analysis for simple examples of 1-dimensional mappings, the Tchebyscheff polynomials of N-th order. Despite the fact that these maps are generally believed to be fully understood since they are semi-conjugated to a Bernoulli shift, the set of non-vanishing higher-order correlations turns out to have a very complicated structure, which is properly described by suitable graphs. These are forests of double N-ary trees. While the case $N = 2$ has been completely solved in a previous paper by one of us [3], we here extend the theory to the case of general N.

The graphs introduced in this paper have many applications. Quite generally they allow for a perturbative evaluation of probability densities of linear combinations of iterates x_n of Tchebyscheff polynomials $T_N(x)$, i.e. of quantities u_M defined by

$$u_M = \tau^{1/2} \sum_{n=1}^{M} c(n) x_n. \tag{1}$$

Here $x_n = T_N(x_{n-1})$, the $c(n)$ are arbitrary coefficients, and $\tau^{1/2}$ is a scale factor. For example, a very simple choice of the coefficients is $c(n) = 1$ and $\tau = M^{-1}$. It is well known that the iterates of the Tchebyscheff maps satisfy a Central Limit Theorem [5]. Hence the density of u_M will become Gaussian for $M \to \infty$. Our graph-theoretical method allows us, among other things, to calculate leading-order corrections to the Gaussian density for large but finite M.

Another interesting application is the coupling of deterministic 'chaotic noise' to a slowly varying evolution dynamics [6]—[9]. We may consider the deterministic chaotic analogue of a linear Langevin equation [10] where the Gaussian white noise is replaced by a chaotic force generated by Tchebyscheff maps or functions of Tchebyscheff maps. One obtains dynamical systems of skew product form [11]—[13]. The coefficients $c(n)$ in eq. (1) then depend on the slowly varying part of the dynamics. Again one can calculate invariant densities in a perturbative way. Each of the graphs introduced in this paper yields a certain contribution in a perturbative series expansion. Physical applications for these types of dynamical systems and their higher-dimensional extensions have been pointed out in hydrodynamics [14, 15], economics [16], quantum mechanics [17] and quantum field theories [18].

This paper is organized as follows. In section 2 we derive diophantic equations whose solutions describe the set of non-vanishing higher-order correlations of Tchebyscheff maps of order N. In section 3 we prove that the solutions can be represented by suitable forests of double N-ary trees. With the help of the graphs we perturbatively calculate invariant densities for skew-product extensions in section 4. Our concluding remarks are given in section 5.

2 Correlations of Tchebyscheff maps

Let us consider iterates $x_{n+1} = T_N(x_n)$ of Tchebyscheff polynomials $T_N(x)$ of order N. The Tchebyscheff polynomials are defined as

$$T_N(x) = \cos(N \arccos x). \tag{2}$$

The first few polynomials are given by

$$\begin{aligned} T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x, \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1. \end{aligned} \tag{3}$$

Let us consider the higher-order correlation functions

$$E(x_{n_1} x_{n_2} ... x_{n_r}) = \int_{-1}^{1} dx_0 \ \rho(x_0) x_{n_1} x_{n_2} ... x_{n_r}. \tag{4}$$

For $N \geq 2$ the maps are semi-conjugated to Bernoulli shifts with an alphabet of N symbols, hence they are ergodic and mixing. The natural invariant density is independent of N and given by

$$\rho(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in [-1, 1]. \tag{5}$$

Defining $x_0 = cos \pi u$ and using $T_N \circ T_M(x) = T_{N \cdot M}(x)$ the iterates can be written as

$$x_n = cos \pi N^n u. \tag{6}$$

Inserting eq. (5) and (6) into eq. (4) we obtain for the correlation functions

$$E(x_{n_1} x_{n_2} ... x_{n_r}) = \int_0^1 du \ cos \pi N^{n_1} u \ldots cos \pi N^{n_r} u, \tag{7}$$

and using $\cos u = \frac{1}{2}(e^{iu} + e^{-iu})$ this can be written as

$$\begin{aligned} E(x_{n_1} x_{n_2} ... x_{n_r}) &= \sum_\sigma \int_0^1 du \prod_{l=1}^{r} \frac{1}{2} exp\,(i \pi \sigma_l N^{n_l} u) \\ &= 2^{-r} \sum_\sigma \int_0^1 du \ exp \left(i \pi u \sum_{l=1}^{r} \sigma_l N^{n_l} \right). \end{aligned} \tag{8}$$

Here $\sum_\sigma$ denotes the summation over all possible configurations $(\sigma_1, ..., \sigma_r)$ with $\sigma_l = \pm 1$. Denoting the Kronecker symbol $\delta_{i,j}$ by $\delta(i,j)$, we obtain

$$E(x_{n_1} x_{n_2} ... x_{n_r}) = 2^{-r} \sum_\sigma \delta(\sigma_1 N^{n_1} + \ldots + \sigma_r N^{n_r}, 0). \tag{9}$$

Apparently, the non-vanishing correlations of the Tchebyscheff maps correspond to tuples $(n_1, \ldots, n_r)$ that are the solutions of the diophantic equations

$$\sum_{i=1}^{r} \sigma_i N^{n_i} = 0, \qquad \sigma_i = \pm 1. \tag{10}$$

These equations will be solved by a graph theoretical method in the next section.

3 Graph theoretical representation of correlations

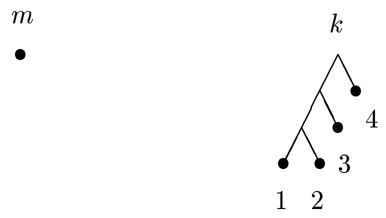


Figure 1: Examples of single binary trees.

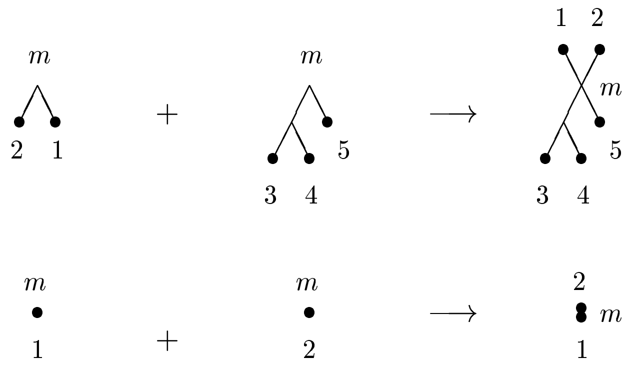
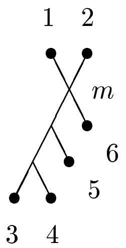


Figure 2: Construction of double binary trees.

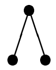


$$= \quad (m-1, m-1; m-3, m-3, m-2, m-1)$$

Figure 3: Identification of a tree with a tuple.

Let us first introduce the relevant graphs for $N = 2$. Later we will generalize to arbitrary N.

Fig. 1 shows some examples of binary trees. The smallest possible tree just consists of a single leaf situated at the root, denoted by m in Fig. 1. The other example in Fig. 1 is a binary tree with root k and leaves $1, 2, 3, 4$. Generally, the trees can be constructed in a recursive way by replacing leaves by forks. Note that these types of trees have various applications in the theory of dynamical systems, for example they have also been shown to characterize the invariant tori of the semi-standard map [19].



$$= \{(m, m-1, m-1), (m-1, m, m-1), (m-1, m-1, m) | m \in \mathbb{N}\}$$

Figure 4: Example of an unlabeled double binary tree.

Figure 5: Example of equivalent trees

To obtain a double binary tree, two single binary trees are put together at their root (Fig. 2). The first example of a double binary tree plotted in Fig. 2 has root m and five leaves in total. We call $1, 2$ upper leaves and $3, 4, 5$ lower leaves. If the single trees consist of one leaf only, then both leaves of the corresponding double tree are directly adjacent to the root. We call these types of trees 'trivial double trees' (second example in Fig. 2).

The trees are identified with tuples of integers. We define a double tree with r leaves and root m to represent a tuple with integer entries given by $(m - d_1, m - d_2, \dots, m - d_r)$. Here d_i denotes the distance of leaf i from the root. An example is shown in Fig. 3. If we don't label the root and the leaves, then the tree stands for the set of tuples that are obtained for $m \in \mathbb{N}$, with the condition of constraint that all components of the tuple should be non-negative. Moreover, all permutations of the components of the tuple are included (Fig. 4). Trees that are obtained by reflections or permutations do not yield anything new and are considered to be equivalent (Fig. 5).

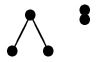
A double binary forest consists of several double binary trees. For the following it is convenient to agree upon the fact that all roots $m_1, \dots, m_s$ of the trees making up the forest correspond to different integer values. The forest represents tuples that are obtained by combining the various tuples represented by the trees into a single tuple, the roots being independent numbers (Fig. 6). Again all possible permutations are included (denoted by P in Fig. 6). The total number of leaves of the forest is the sum of the number of leaves of the constituent trees.

In [3] it was proved that the set of tuples $(n_1, \dots, n_r)$ for which the r-point correlation function $E(x_{n_1} \cdots x_{n_r})$ of the map $T_2(x) = 2x^2 - 1$ does not vanish is given by the union $L_r^{(2)}$ of all double binary forests with r leaves. The first few sets $L_r^{(2)}$ are shown in Fig. 7.

While a complete proof of this statement can be found in [3], let us here just sketch the proof. Clearly, by construction the forests are solutions of at least one of the diophantic equations (10) for $N = 2$. For example, the forest $L_2^{(2)}$ represents solutions of the form $(n_1, n_2) = (m, m)$ of the equation $2^{n_1} = 2^{n_2}$. The forest $L_3^{(2)}$ represents solutions of the form $(n_1, n_2, n_3) = (m, m-1, m-1)$ of the equation $2^{n_1} = 2^{n_2} + 2^{n_3}$ (plus all possible permutations). Generally, the α upper leaves of a forest correspond to the terms on the left hand side of the equation

$$2^{n_1} + \dots 2^{n_\alpha} = 2^{n_{\alpha+1}} + \dots + 2^{n_r}, \tag{11}$$

the $r - \alpha$ lower leaves to the terms on the right hand side. Without restriction of generality we assume that the number of upper leaves is smaller or equal to



$$= \quad \{P(m_1, m_2; m_1 - 1, m_1 - 1, m_2)|m_1, m_2 \in \mathbb{N}, m_1 \neq m_2)\}$$

Figure 6: Example of an unlabeled double binary forest

the number of lower leaves. If, proceeding from $L_r^{(2)}$ to $L_{r+1}^{(2)}$, a leaf is replaced by a fork, then clearly the new forest also represents a solution of (10), with r increased by 1, since $2^n = 2^{n-1} + 2^{n-1}$. One also has to show that the forests indeed represent all possible solutions of the diophantic equations. This follows from the uniqueness of the binary representation [3].

Let us now generalize for Tchebyscheff maps of arbitrary order N. We have to find all solutions $(n_1, \ldots, n_r)$ of the diophantic equations

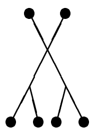
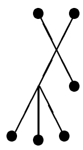
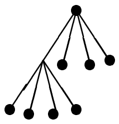
$$N^{n_1} + \ldots + N^{n_\alpha} = N^{n_{\alpha+1}} + \ldots + N^{n_r}. \tag{12}$$

For general N more general types of graphs are relevant, which we call double N-ary forests. They are constructed as follows. First, consider the following special types of (single) trees.

We call them k-forks, k being the number of leaves. Secondly, consider the following symmetric double trees consisting of two k-forks.

We call them basis trees. A basis tree is called a k-basis tree if it consists of two k-forks. Formally, the trivial double tree can be regarded as a 1-basis tree (consisting of two 1-forks).

For a given Tchebyscheff map of order N, we now construct double N-ary trees by taking any of the k-basis trees with $k = 1, 2, \ldots, N-1$ and replacing leaves by N-forks. Examples are

a)  b)  c) 

The figure shows some double N-ary trees for $N = 2$ (a), $N = 3$ (b), $N = 4$ (c). To construct these trees we started from a 1-basis tree (the trivial double tree) in example a) and subsequently replaced leaves by 2-forks (four times in total). In b) we started from a 2-basis tree and replaced one leaf by a 3-fork. In c) we started from a 1-basis tree and subsequently replaced two leaves by

4-forks. Generally, for e.g. $N = 4$ we can start from a 1,2- or 3-basis tree and replace leaves by 4-forks arbitrary often.

Double N-ary forests consist of several double N-ary trees. We identify them with sets of tuples in just the same way as we described before, i.e. the distances of the leaves from the root determine the structure of the tuple, and the roots are independent integers. The union of all double N-ary forests with r leaves is denoted by $L_r^{(N)}$. These sets are plotted in Fig. 7–10 for various N. We now have the following

Theorem.
Let x_n denote the iterates of the N-th order Tchebyscheff polynomial. Then

$$E(x_{n_1} \cdots x_{n_r}) \neq 0 \Longleftrightarrow (n_1, \ldots, n_r) \in L_r^{(N)}. \tag{13}$$

Proof.
i) "$\Longleftarrow$" Let $(n_1, \ldots, n_r) \in L_r^{(N)}$. This means that the tuple can be represented as a double N-ary forest. Let $n_{i_1}, \ldots, n_{i_\alpha}$ be the top leaves and $n_{i_{\alpha+1}}, \ldots, n_{i_{\alpha+\beta}}$ be the bottom leaves ($\alpha + \beta = r$). Proceeding along an N-fork towards the root can be regarded as performing the operation $N^n = N \cdot N^{n-1}$. After a while, doing this for both upper and lower leaves, we arrive at the symmetric double trees at the roots of the trees. We obtain $N^{n_{i_1}} + \cdots + N^{n_{i_\alpha}} = \xi_1 N^{m_1} + \cdots + \xi_s N^{m_s} = N^{n_{i_{\alpha+1}}} + \cdots + N^{n_{i_{\alpha+\beta}}}$, where the numbers ξ_i characterize the ξ_i-basis trees. It follows that at least one of the Kronecker deltas in eq. (9) has the value $+1$, and therefore $E(x_{n_1} \cdots x_{n_r}) \neq 0$.

ii) "$\Longrightarrow$" Let $E(x_{n_1} \cdots x_{n_r}) \neq 0$. Then $(n_1, \ldots n_r)$ is a solution of at least one of the equations $\sigma_1 N^{n_1} + \cdots + \sigma_r N^{n_r} = 0$, $\sigma_i = \pm 1$. Without loss of generality we assume that $(n_1, \ldots, n_r)$ satisfies $N^{n_1} + \cdots + N^{n_\alpha} = N^{n_{\alpha+1}} + \cdots + N^{n_{\alpha+\beta}}$, $\alpha + \beta = r, \alpha \leq \beta$. Define the integer k by $k = N^{n_1} + \cdots + N^{n_\alpha}$. The numbers $n_1, \ldots, n_\alpha$ need not to be different, but there is a unique representation of the integer k in the N-system. We can write $k = \sum_{i=0}^{\infty} \xi_i N^i$, where the integers ξ_i take on values in $\{0, 1, \cdots, N-1\}$. Equivalently, we can write $k = \sum_{i=1}^{s} \xi_i N^{m_i}$ with $\xi_i \in \{1, 2, \ldots, N-1\}$, and all $m_i \in \mathbb{N}_0$ are different. Now given the numbers $n_1, \ldots, n_\alpha$ we check if N of them are equal. If yes, we replace them by a new integer that is larger by 1 and draw an N-fork. We continue in the same way with the new set of numbers, until we don't find any N equal numbers any more. The same is done with the numbers $n_{\alpha+1}, \cdots, n_r$. By this method we finally arrive at s groups of different numbers, each group containing ξ_i equal numbers. The ξ_i describe the unique representation of the number k in the N-system. Due to the uniqueness of this representation we obtain the same ξ_i starting either from $n_1, \ldots, n_\alpha$ or $n_{\alpha+1}, \ldots, n_r$. Thus we can draw the symmetric ξ_i-basis trees and connect upper and lower trees of the forest. Hence, $(n_1, \cdots, n_r)$ can be represented as a double N-ary forest, q.e.d.

4 Perturbation theory

4.1 Linear combinations of iterates

We will now deal with linear combinations u_M defined by

$$u_M = \tau^{\frac{1}{2}} \sum_{n=1}^{M} c(n) x_n. \tag{14}$$

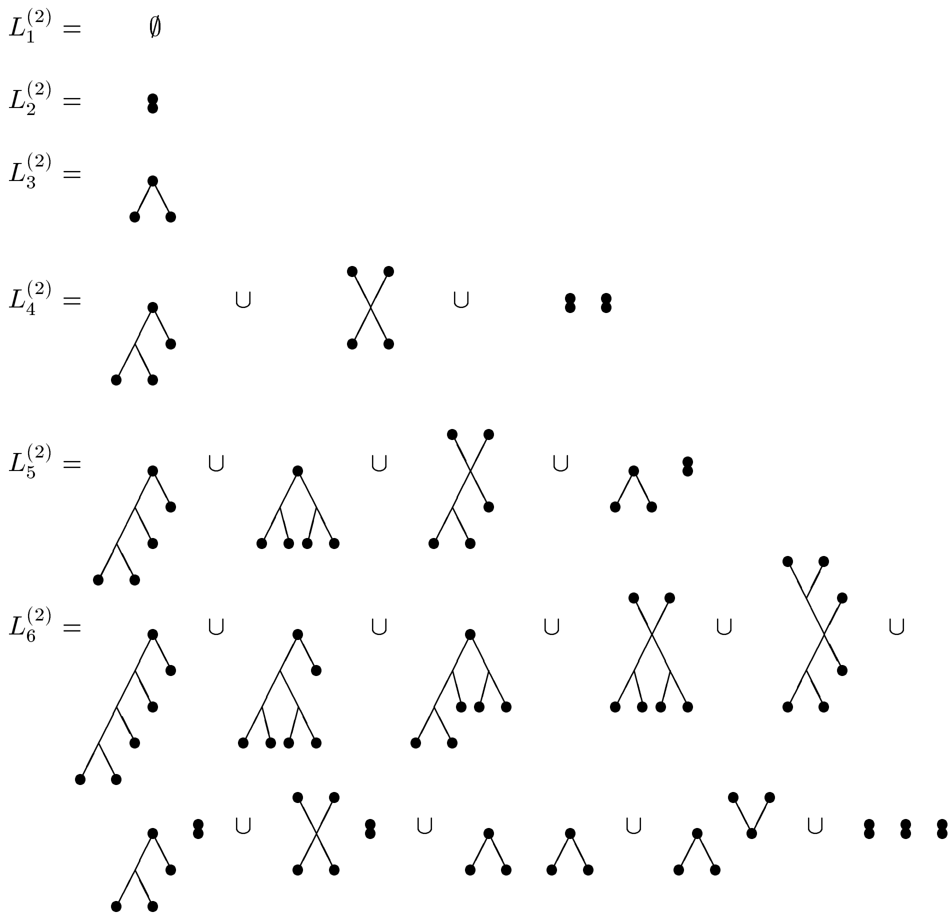


Figure 7: Forests describing the correlations of $T_2(x)$.

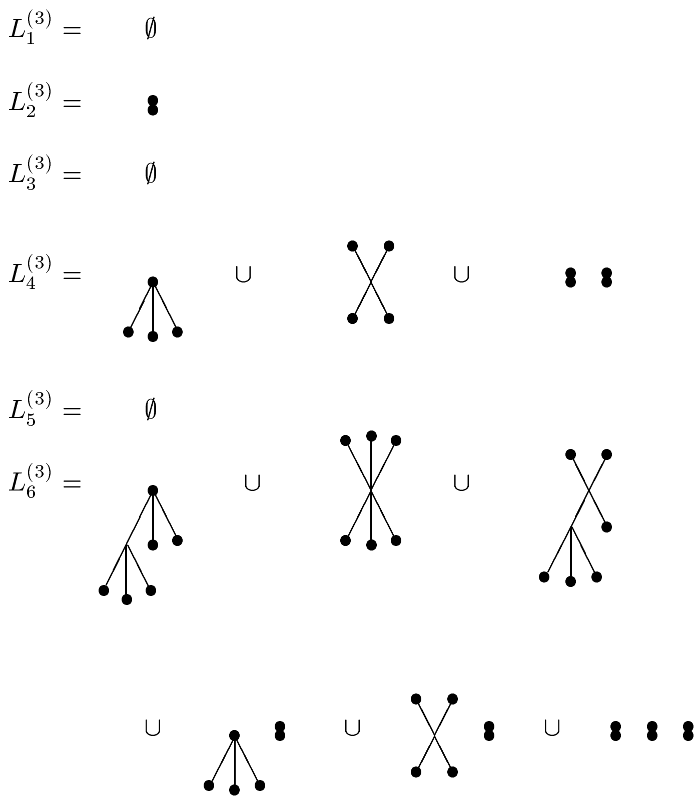


Figure 8: Forests describing the correlations of $T_3(x)$.

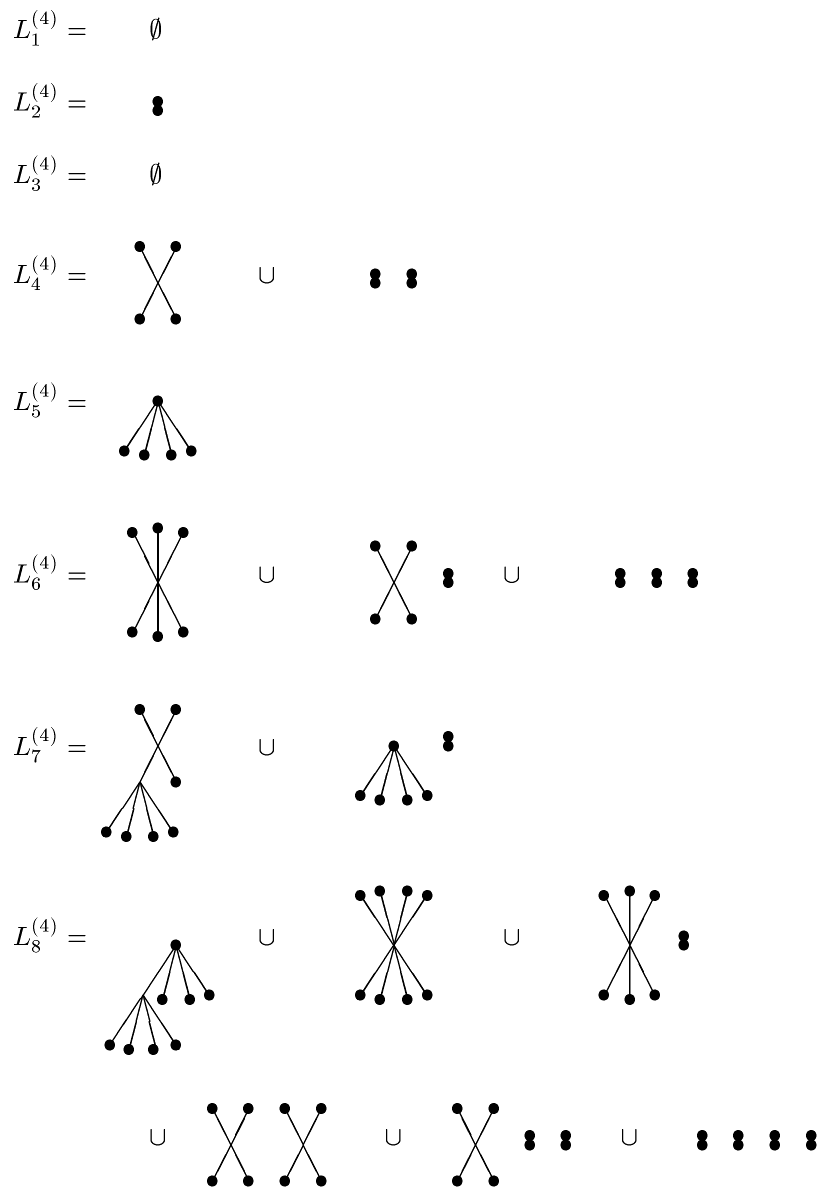


Figure 9: Forests describing the correlations of $T_4(x)$.

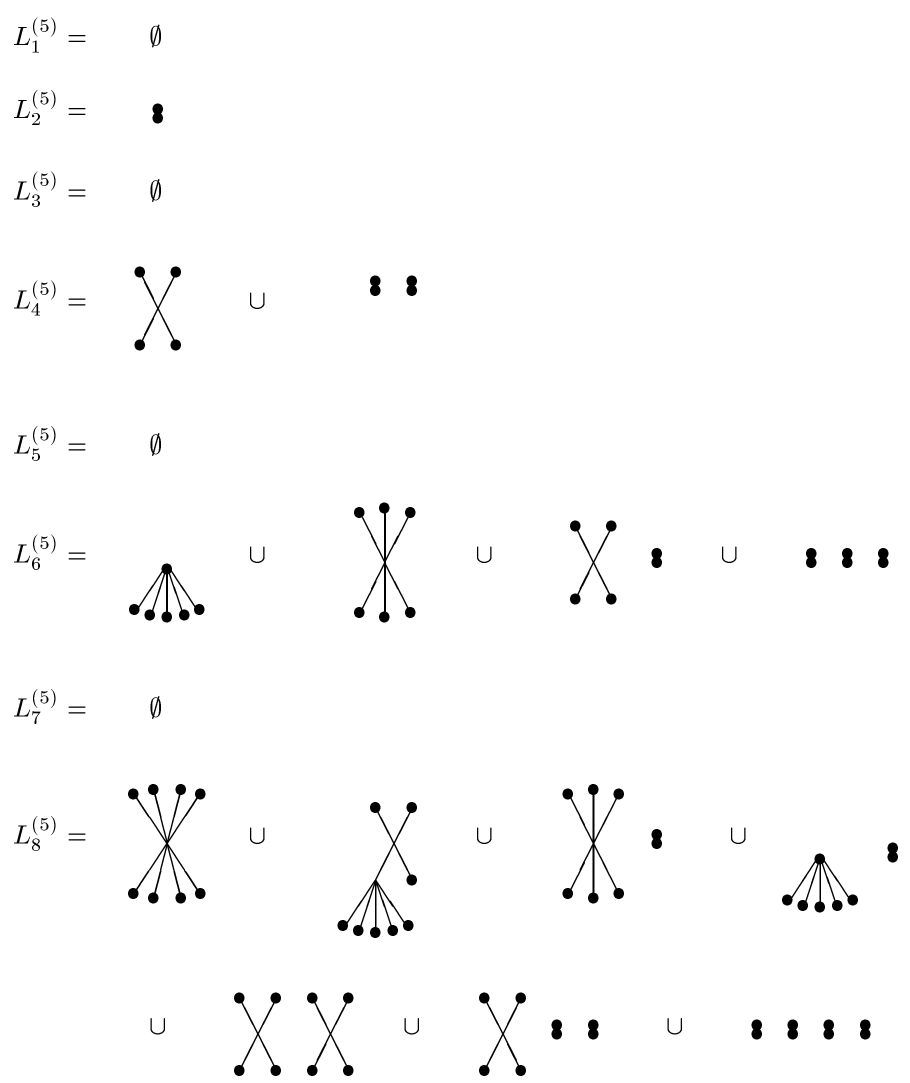


Figure 10: Forests describing the correlations of $T_5(x)$.

First, let us give some physical motivation why such a linear combination of iterates is relevant.

Consider a linear Langevin equation where the Gaussian white noise is replaced by deterministic 'chaotic noise' (kicks of strength x_n on a small time scale τ).

$$\dot{u} = -\gamma u + \sqrt{\tau}\sum_{n=1}^{\infty} x_n \delta(t - n\tau). \tag{15}$$

The x_n are assumed to evolve according to some chaotic map, for example a Tchebyscheff map. Integrating the equation of motion one obtains the following recurrence relation of skew product type

$$\begin{aligned} x_{n+1} &= T(x_n) \\ u_{n+1} &= \lambda u_n + \sqrt{\tau} x_{n+1}, \qquad \lambda = e^{-\gamma\tau} \end{aligned} \tag{16}$$

with $u_n := u(n\tau + 0)$. Iterating this M times one sees that the variable u_M can be written in the form (14). The coefficients are given by $c(n) = \lambda^{M-n} = e^{-\tau(M-n)}$ (for simplicity we assume $u_0 = 0$ and $\gamma = 1$). Much more details on these and more general types of systems can be found in [6]. Higher-dimensional cascade-like extensions of eq. (16) can serve as models for hydrodynamic [14, 15] and economic [16] turbulence.

The correlation functions $E(u_{M_1}\cdots u_{M_r})$ are superpositions of the $E(x_{n_1}\cdots x_{n_r})$

$$E(u_{M_1}\cdots u_{M_r}) = \tau^{\frac{r}{2}} \sum_{n_1=1}^{M_1} \cdots \sum_{n_r=1}^{M_r} c(n_1)\ldots c(n_r) E(x_{n_1}\cdots x_{n_r}). \tag{17}$$

The right hand side of eq. (17) can be further evaluated for $T = T_N$ by adding up the contributions of the forests described in the previous section. We insert (9) into (17) to obtain

$$\begin{aligned} E(u_{M_1}\cdots u_{M_r}) &= 2^{-r}\tau^{\frac{r}{2}} \sum_{n_1=1}^{M_1} \cdots \sum_{n_r=1}^{M_r} c(n_1)\ldots c(n_r) \\ &\quad \cdot \sum_{\sigma} \delta(\sigma_1 N^{n_1} + \ldots \sigma_r N^{n_r}, 0) \end{aligned} \tag{18}$$

The moments $E(u_M^r)$ $(M_1 = M_2 = \ldots = M_r = M)$ are of particular interest. If we again denote the number of positive σ_i by α, then there are $\binom{r}{\alpha}$ diophantic equations of that type and we obtain

$$\begin{aligned} E(u_M^r) &= 2^{-r}\tau^{\frac{r}{2}} \sum_{\alpha=1}^{r-1} \binom{r}{\alpha} \sum_{n_1=1}^{M} \cdots \sum_{n_r=1}^{M} c(n_1)\ldots c(n_r) \\ &\quad \cdot \delta(N^{n_1} + \ldots + N^{n_\alpha}, N^{n_{\alpha+1}} + \ldots + N^{n_r}). \end{aligned} \tag{19}$$

If we have the moments we can evaluate the characteristic function

$$G(k) := E(e^{iku_M}) = \sum_{r=0}^{\infty} \frac{(ik)^r}{r!} E(u_M^r). \tag{20}$$

The probability density of u (for initial values x_1 distributed according to (5)) follows by Fourier transformation,

$$p(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, G(k) e^{-iku}. \tag{21}$$

Note that only forests with an odd number of leaves lead to odd powers of k in the characteristic function and hence to odd contributions for the invariant density of u.

We may now regard τ as a small parameter and apply perturbation theory. For a forest that contains s trees, the r sums in (19) are reduced to s sums, since only the roots represent independent integers. Since $M \sim t/\tau$, where t is the total time elapsed, each sum is of the order τ^{-1}. Due to the factor $\tau^{\frac{r}{2}}$ a forest yields a total contribution of order $\tau^{\frac{r}{2}-s}$. The more trees a forest contains, the lower is the order in τ and the larger the contribution. The largest contribution comes from $s = r/2$, i.e. the case where each tree in the forest is a trivial double tree. Forests with an odd number of leaves lead to contributions of order $\tau^{(2j-1)/2}$, trees with an even number of leaves to contributions of order $\tau^{j-1}, j \in \mathbb{N}$.

For the binary forests shown in Fig. 7 the contributions are as follows. In the set of forests with two leaves, $L_2^{(2)}$, there is only one trivial double tree which leads to a contribution of order 1. For $L_3^{(2)}$, the forest with three leaves gives a contribution of order $\tau^{1/2}$. The first two forests in $L_4^{(2)}$ each consist of just one tree, so that their contribution is of order τ. The trivial double trees again yield a contribution of order 1. For $L_5^{(2)}$ there are three forests consisting of one tree and one forest consisting of two trees, so that the orders are $\tau^{3/2}$ and $\tau^{1/2}$, respectively. $L_6^{(2)}$ yields contributions of order τ^2, τ and 1.

For the 3-ary forests describing the correlations of the Tchebyscheff polynomial of 3rd order (Fig. 8), the contributions are of order 1 in $L_2^{(3)}$, of order τ and 1 in $L_4^{(3)}$ and of order τ^2, τ and 1 in $L_6^{(3)}$. Since there are no forests with an odd number of leaves, the density $p(u)$ is an even function, which is in agreement with previous results concerning the symmetry of the dynamics [6].

For $N = 4$ (Fig. 9), $L_5^{(4)}$ and $L_7^{(4)}$ are not empty but contain forests that yield contributions of order $\tau^{3/2}$ and $\tau^{5/2}$. Hence, due to the odd number of leaves the invariant density is not symmetric any more. In general, the first odd contribution for the polynomial of even order N, $N \geq 2$, is from forests with $N+1$ leaves that have only one top and N bottom leaves (the double tree version of an N-fork). The corresponding contribution to the invariant density is of order $\tau^{(N-1)/2}$.

In the following sections we will calculate the various contributions explicitly up to order τ.

4.2 The Gaussian limit case

Contributions of order 1 correspond to forests which contain only trivial double trees. These forests are the same for all Tchebyscheff polynomials. In the limit of vanishing τ, the invariant density is therefore the same for all polynomials which is in agreement with the expectation based on previous calculations and simulations [5, 7, 9].

The $s = r/2$ trivial double trees represent tuples of the form

$$\vdots \;\; \vdots \;\; \ldots \;\; \vdots \; : \qquad (n_1, \ldots n_r) = (m_1, \ldots m_s; m_1, \ldots m_s).$$

The $\left(\begin{array}{c} r \\ \alpha \end{array} \right)$ possibilities to chose s upper leaves form the total number of leaves r has already been taken into account in (19). In addition, there are $s!$ permutations of the lower leaves. The contribution of order τ^0 is

$$C_0^N(s) = 2^{-r}\tau^{\frac{r}{2}} \begin{pmatrix} r \\ s \end{pmatrix} s! \underbrace{\sum_{m_1=1}^{M} \cdots \sum_{m_s=1}^{M}}_{m_i \text{ different}} c^2(m_1)\ldots c^2(m_s). \tag{22}$$

Here and in the following we denote the contribution of order τ^a for the Tchebyscheff polynomial of order N as C_a^N. In (22) the sum over the leaves n_i has been replaced by a sum over the roots m_i. The m_i must be different, otherwise we can't use the uniqueness of the N-ary representation in our theorem.

If we now assume that the m_i are not necessarily different, we only make an error of order τ. This error will be considered in section 4.4 when contributions of order τ will be calculated. We thus obtain

$$C_0^N(s) = 2^{-2s}\tau^s \frac{(2s)!}{s!} \left(\sum_{m=1}^{M} c^2(m)\right)^s + O(\tau) \tag{23}$$

and for the characteristic function in 0-th order

$$\begin{aligned} G_0^N(k) &= \sum_{s=0}^{\infty} \frac{i^{2s}}{(2s)!} k^{2s} C_0^N(s) \\ &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \tau^s 2^{-2s} \left(\sum_{m=1}^{M} c^2(m)\right)^s k^{2s} = e^{-\beta_1 k^2}, \end{aligned}$$

$$\beta_1 := \frac{1}{4}\tau \sum_{m=1}^{M} c^2(m). \tag{24}$$

Thus in 0-th order the invariant density of u is a Gaussian

$$\begin{aligned} p_0^N(u) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, G_0(k) e^{-iku} \\ &= \frac{1}{\sqrt{4\pi\beta_1}} \exp\left\{-\frac{u^2}{4\beta_1}\right\} \end{aligned} \tag{25}$$

for all Tchebyscheff polynomials ($\tau \to 0, M \to \infty, t = M\tau$ finite). This is just another proof of the known fact that the Tchebyscheff maps satisfy a Central Limit Theorem for $\tau \to 0$. Inserting, as our main example, the coefficients $c(m) = \lambda^{M-m}$ of the linear relaxation dynamics we obtain

$$\beta_1 = \frac{1}{4}\tau \frac{1-\lambda^{2M}}{1-\lambda^2}. \tag{26}$$

For a large number of iteration steps $M \to \infty$ it follows

$$\beta_1 = \frac{1}{8} + \frac{1}{8}\tau + O(\tau^2), \tag{27}$$

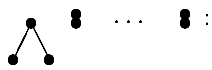
where $\lambda = e^{-\gamma\tau}$ and $\gamma = 1$ have been used. Inserting β_1 into (25) we finally obtain

$$p_0^N(u) = \sqrt{\frac{2}{\pi}} e^{-2u^2}, \tag{28}$$

a Gaussian with standard deviation $\langle u^2 \rangle^{\frac{1}{2}} = 1/2$.

4.3 Corrections of order $\tau^{\frac{1}{2}}$

In order to give a contribution of order $\tau^{\frac{1}{2}}$ a forest must consist either of one tree with three leaves or of trivial trees and one tree with three leaves. The number of trees is then $s = (r-1)/2$. This case only appears for binary trees that describe the correlations of the polynomial $T_2(x)$



$$: \quad (n_1, \ldots n_r) = (m_1, \ldots m_s; m_1 - 1, m_1 - 1, m_1, m_2, \ldots m_s).$$

For the Tchebyscheff polynomials of order $N > 2$ the equations $N^{n_1} = N^{n_2} + N^{n_3}$ have no solutions and hence there is no correction of order $\tau^{\frac{1}{2}}$ to the Gaussian. The case $N = 2$ has been completely solved in [3], so here we just quote the final result for the density including terms of order $\tau^{1/2}$, which was derived from the above graph.

$$p_{1/2}^{N=2}(u) = \frac{1}{\sqrt{4\pi\beta_1}} \left(1 + \tau^{1/2} \left(\frac{\beta_4}{8\beta_1^3}(u^3 - 6\beta_1 u)\right)\right) \exp\left\{-\frac{u^2}{4\beta_1}\right\} \quad (29)$$

$$\beta_4 := \frac{1}{8}\tau \sum_{m=2}^{M} c(m)c^2(m-1). \quad (30)$$

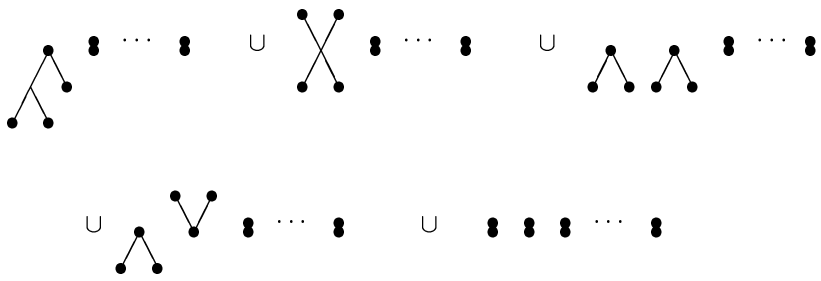
For our standard example of coefficients $c(m)$ this yields for $M \to \infty$

$$p_{1/2}^{N=2}(u) = \sqrt{\frac{2}{\pi}} \left\{1 + \tau^{1/2}(-2u + \frac{8}{3}u^3)\right\} e^{-2u^2}. \quad (31)$$

Note that the first correction to the Gaussian is an odd function. This leads to an asymmetric invariant density for finite values of τ. This is illustrated in Fig. 11. The figure shows numerical results obtained by iterating the map (16) and producing a histogram of the variable u_n.

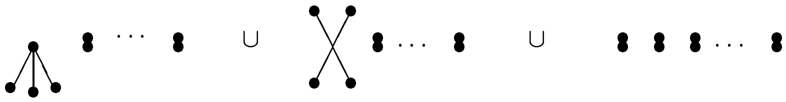
4.4 Corrections of order τ

The forests that lead to corrections of order τ have $s = r/2 - 1$ trees. For the Tchebyscheff polynomial of second order the corresponding forests are



In section 4.2 eq. (23) has been obtained by assuming that the summation indices do not differ. The error made due to this assumption is of order τ. Since we have to consider this contribution here the trivial forests have been listed once again.

For the 3rd-order Tchebyscheff polynomial contributions of order τ originate from the forests



and for the polynomials of 4th and higher order they originate from

only. These forests of order τ are the same for all polynomials of 4th- and higher order, while for the 3rd-order polynomial the additional type of forest must be considered as well. This explains why the corrections to the Gaussian density are practically the same for $T_4(x)$, $T_5(x)$ and $T_6(x)$ (Fig. 12), while those for the $T_3(x)$, shown in Fig. 11b, are slightly different. The forests describing correlations of $T_2(x)$ differ even more.

We will first deal with the two forests that describe contributions of order τ for the Tchebyscheff polynomials of fourth and higher order since these contributions also appear for $T_2(x)$ and $T_3(x)$.

The first forest to be evaluated is

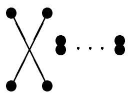
 $: \quad (n_1, \ldots n_r) = (m_1, \ldots m_s; m_1, \ldots m_s).$

Its contribution is

$$
\begin{aligned}
& 2^{-r}\tau^s \binom{r}{s} s! \underbrace{\sum_{m_1=1}^{M} \cdots \sum_{m_s=1}^{M}}_{m_1\ldots m_r \text{different}} c^2(m_1)\ldots c^2(m_s) \\
= \; & 2^{-2s}\tau^s \binom{2s}{s} s! \\
& \cdot \left\{ \left(\sum_{m=1}^{M} c^2(m)\right)^s - \binom{s}{2} \left(\sum_{m=1}^{M} c^4(m)\right) \left(\sum_{m=1}^{M} c^2(m)\right)^{s-2} \right\} + O(\tau^2) \\
= \; & 2^{-2s}\tau^s \frac{(2s)!}{s!} \left(\sum_{m=1}^{M} c^2(m)\right)^s \\
& -2^{-2s-1}\tau^s \frac{2s!}{(s-2)!} \left(\sum_{m=1}^{M} c^4(m)\right) \left(\sum_{m=1}^{M} c^2(m)\right)^{s-2} + O(\tau^2). \qquad (32)
\end{aligned}
$$

The first term on the right hand side of (32) is the contribution of order 1 which has already been calculated in (23). The second term is a correction of order τ due to the fact that all m_i must be different.

Next, we evaluate the contribution of

 $: (n_1, \ldots n_r) = (m_1 - 1, m_1 - 1, m_2, \ldots, m_{s-1}; m_1 - 1, m_1 - 1, m_2, \ldots, m_{s-1}).$

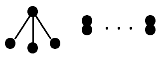
There are $\left(\begin{array}{c} s \\ 2 \end{array} \right)$ possibilities of choosing two leaves from the s upper leaves and $s!/2!$ permutations of the lower leaves. Hence we obtain the contribution

$$\begin{aligned} & 2^{-r}\tau^{s}\left(\begin{array}{c} r \\ s \end{array} \right)\left(\begin{array}{c} s \\ 2 \end{array} \right)\frac{s!}{2!}\left(\sum_{m=2}^{M+1} c^{4}(m-1)\right)\left(\sum_{m=1}^{M} c^{2}(m)\right)^{s-2} \\ = \;& 2^{-2s-2}\tau^{s}\frac{(2s)!}{(s-2)!}\left(\sum_{m=1}^{M} c^{4}(m)\right)\left(\sum_{m=1}^{M} c^{2}(m)\right)^{s-2} + O(\tau^{2}). \qquad (33) \end{aligned}$$

The total contribution of order τ for the Tchebyscheff polynomials of fourth and higher order is the sum of (33) and the second term in (32)

$$C_1^{N\geq 4}(s) = -2^{-2s-2}\tau^{s}\frac{(2s)!}{(s-2)!}\left(\sum_{m=1}^{M} c^{4}(m)\right)\left(\sum_{m=1}^{M} c^{2}(m)\right)^{s-2}. \qquad (34)$$

In order to obtain the $O(\tau)$ contributions for the third Tchebyscheff polynomial, the contribution of the forests

 $: (n_1,\ldots n_r) = (m_1,\ldots m_s; m_1-1, m_1-1, m_1-1, m_2,\ldots m_s)$

has to be evaluated as well. Now the number of trees is $s = r/2-1$. For given $m_1,\ldots,m_s$ there are $(s+2)!/3!$ permutations of the lower leaves. In addition there are s possibilities of choosing an m_i that corresponds to the 3-fork. For all forests that have a different number of top and bottom leaves the number of allowed forests has to be multiplied by a factor 2, which takes into account trees for which upper and lower parts are exchanged. We obtain

$$\begin{aligned} C_1^{3+}(s) &= 2^{-r+1}\tau^{r/2}\left(\begin{array}{c} r \\ s \end{array} \right)\frac{(s+2)!}{3!}s\left(\sum_{m=2}^{M} c(m)c^{3}(m-1)\right)\left(\sum_{m=1}^{M} c^{2}(m)\right)^{s-1} \\ &= 2^{-2s-1}\tau^{s+1}\frac{(2s+2)!}{6(s-1)!} \\ &\quad \cdot\left(\sum_{m=2}^{M} c(m)c^{3}(m-1)\right)\left(\sum_{m=1}^{M} c^{2}(m)\right)^{s-1}, \qquad (35) \end{aligned}$$

which has to be added to (34).

For $N \geq 4$ the characteristic function up to order τ is obtained as

$$\begin{aligned} G_1^{N\geq 4}(k) &= G_0(k) + \sum_{s=2}^{\infty}\frac{i^{2s}}{(2s)!}k^{2s}C_1^{(N\geq 4)}(s) \\ &= (1-\tau\beta_2 k^4)e^{-\beta_1 k^2} \qquad (36) \end{aligned}$$

with

$$\beta_2 := \frac{1}{64}\tau\sum_{m=1}^{M} c^{4}(m). \qquad (37)$$

For $N=3$ we also have to add the contribution $C_1^{3+}(s)$, and we obtain

$$G_1^{3}(k) = (1-\tau\beta_2 k^4 + \tau\beta_3 k^4)e^{-\beta_1 k^2} \qquad (38)$$

with

$$\beta_3 := \frac{1}{48}\tau \sum_{m=2}^{M} c(m)c^3(m-1). \tag{39}$$

By Fourier transformation of $G_1^{N\geq 4}$ and G_1^3 we get [20]

$$\begin{aligned} p_1^{N\geq 4}(u) &= \frac{1}{\sqrt{4\pi\beta_1}}\Big(1-\tau\frac{\beta_2}{2^4\beta_1^2}\Big(\frac{1}{\beta_1^2}u^4-12\frac{1}{\beta_1}u^2+12\Big)\Big)\exp\left\{-\frac{u^2}{4\beta_1}\right\} \\ p_1^3(u) &= \frac{1}{\sqrt{4\pi\beta_1}}\Big(1+\tau\frac{(\beta_3-\beta_2)}{2^4\beta_1^2}\Big(\frac{1}{\beta_1^2}u^4-12\frac{1}{\beta_1}u^2+12\Big)\Big)\exp\left\{-\frac{u^2}{4\beta_1}\right\}. \end{aligned} \tag{40}$$

For completeness we also list the result for the second Tchebyscheff polynomial that has been derived in [3]

$$\begin{aligned} p_1^2(u) = & \frac{1}{\sqrt{4\pi\beta_1}}\Big(1+\tau^{1/2}\Big(\frac{\beta_4}{8\beta_1^3}(u^3-6\beta_1 u)\Big) \\ &+ \tau\Big(\big(\frac{(3\beta_5-\beta_2)}{4\beta_1^2}-\frac{15\beta_4^2}{16\beta_1^3}\big)+\big(\frac{45\beta_4^2}{32\beta_1^4}-\frac{3(\beta_5-\beta_2)}{4\beta_1^3}\big)u^2 \\ &+ \big(\frac{\beta_3-\beta_2}{16\beta_1^4}-\frac{15\beta_4^2}{64\beta_1^5}\big)u^4+\frac{\beta_4^2}{128\beta_1^6}u^6\Big)\Big)\exp\left\{-\frac{u^2}{4\beta_1}\right\} \end{aligned} \tag{41}$$

with

$$\beta_4 := \frac{1}{8}\tau\sum_{m=2}^{M} c(m)c^2(m-1), \qquad \beta_5 := \frac{1}{16}\tau\sum_{m=3}^{M} c(m)c(m-1)c^2(m-2). \tag{42}$$

Again we may further evaluate this for the example of a damped particle driven by chaotic forces, where $c(m)=\lambda^{M-m}$ and $\lambda=e^{-\tau}$. Inserting this into (37), (39) and (42) we get in the limit $M\to\infty$

$$\begin{aligned} \beta_2 &= \frac{1}{256}+\frac{1}{128}\tau+O(\tau^2) \\ \beta_3 &= \frac{1}{192}-\frac{1}{192}\tau+O(\tau^2) \\ \beta_4 &= \frac{1}{24}-\frac{1}{48}\tau+O(\tau^2) \\ \beta_5 &= \frac{1}{64}-\frac{3}{64}\tau+O(\tau^2). \end{aligned}$$

With β_1 from eq. (27) we finally obtain for the invariant density up to order τ

$$p_1^{N\geq 4}(u) = \sqrt{\frac{2}{\pi}}\left(1+\tau(-u^4+\frac{7}{2}u^2-\frac{11}{16})\right)e^{-2u^2} \quad \text{for} \quad N\geq 4, \tag{43}$$

$$p_1^3(u) = \sqrt{\frac{2}{\pi}}\left(1+\tau(\frac{1}{3}u^4+\frac{3}{2}u^2-\frac{7}{16})\right)e^{-2u^2} \quad \text{for} \quad N=3 \tag{44}$$

$$\begin{aligned} p_1^2(u) = & \sqrt{\frac{2}{\pi}}\Big(1+\tau^{\frac{1}{2}}(-2u+\frac{8}{3}u^3) \\ &+\tau(\frac{32}{9}u^6-\frac{31}{3}u^4+\frac{15}{2}u^2-\frac{37}{48})\Big)e^{-2u^2} \quad \text{for} \quad N=2. \end{aligned} \tag{45}$$

In Fig. 13 and 14 our analytical results for $T_3(x)$ and $T_N(x)$, $N \geq 4$ are compared with numerical results obtained by iterating the map (16) and calculating histograms. Again we have plotted the difference to the Gaussian obtained in the limit $\tau \to 0$. The agreement between numerical and analytical results is very good.

Fig. 15 shows the sum of the corrections of order $\tau^{1/2}$ and τ for T_2 in comparison with the numerical result. The agreement is good but some differences remain, in particular near the extrema of the distribution. These deviations are caused by the neglected contributions of order $\tau^{3/2}$, which appear to be larger than the neglected contributions of order τ^2 for $T_3(x)$ and of order $\tau^{3/2}$ for $T_4(x)$.

The symmetry properties of the invariant densities can be explained by the topological structure of the graphs. For $T_2(x)$, the correction of order $\tau^{\frac{1}{2}}$ dominates the deviations from Gaussian behaviour and causes a clear asymmetry of the invariant density. For the polynomials of higher order, the first corrections are of order τ and they yield a symmetric contribution to the invariant density. For all Tchebyscheff polynomials with odd N all odd moments of u vanish so that the invariant density is exactly symmetric in all orders of perturbation theory. For even N, the first odd contribution is of order $\tau^{(N-1)/2}$, so for large N and small τ the densities are approximately symmetric.

5 Conclusion

In this paper, in a sense, we have analyzed the properties of 'chaotic noise'. We have determined the hierarchy of all correlation functions for a deterministic dynamics generated by Tchebyscheff polynomials of order N. Whereas the higher-order correlation functions of Gaussian white noise are trivial, i.e. they are obtained as suitable products of 2-point functions, the set of non-vanishing higher-order correlations of the Tchebyscheff maps has a very interesting and non-trivial self-similar structure. We introduced suitable graphs that describe this structure. We proved that the relevant graphs are forests of double N-ary trees.

Our method allows for a perturbative treatment of Langevin-like equations where the Gaussian white noise is replaced by a deterministic chaotic dynamics on a small time scale. Generally, of course, more general types of chaotic forces can be thought of. The systems studied here are simple examples where an analytic treatment is possible. Even for these 'simple' examples the structure of higher-order correlations is very complicated, and interesting connections with graph and number theory arise.

References

[1] I.P. Cornfeld, S.V. Fomin, and Ya Sinai, *Ergodic Theory*, Springer, New York (1982)

[2] T. Geisel and V. Fairen, Phys. Lett. **105A**, 263 (1984)

[3] C. Beck, Nonlinearity **4**, 1131 (1991)

[4] C. Nicolis and G. Nicolis, Phys. Rev. **58E**, 4391 (1998)

[5] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York (1968)

[6] C. Beck, Physica **233A**, 419 (1996)

[7] C. Beck and G. Roepstorff, Physica **145A**, 1 (1987)

[8] T. Shimizu, Physica **164A**, 123 (1990)

[9] A. Hilgers and C. Beck, Phys. Rev. **60E**, 5385 (1999)

[10] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North-Holland, Amsterdam (1981)

[11] J.L. Kaplan and J.A. Yorke, in *Functional differential equations and the approximation of fixed points*, Lecture Notes in Mathematics, vol. 730, p.228 (1979)

[12] W. Parry and M. Pollicott, J. London Math. Soc. **56**, 395 (1997)

[13] C.P. Walkden, Commun. Math. Phys. **205**, 263 (1999)

[14] C. Beck, Phys. Rev. **49E**, 3641 (1994)

[15] A. Hilgers and C. Beck, Europhys. Lett. **45**, 552 (1999)

[16] A. Hilgers and C. Beck, Int. Journ. Bif. Chaos **7**, 1855 (1997)

[17] C. Beck, Physica **85D**, 459 (1995)

[18] C. Beck, Nonlinearity **8**, 423 (1995)

[19] I. Percival and F. Vivaldi, Physica **33D**, 304 (1988)

[20] I. S. Gradshteyn und I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York (1965)

a) $p(u)$

b) $p(u) - p_0(u)$

T_2
T_3
T_4

T_2
T_3
T_4

u

u

Figure 11: a) Invariant density $p(u)$ for the system (16); b) deviation of $p(u)$ from the Gaussian $p_0(u)$ for the second, third and fourth Tchebyscheff-polynomial, $\tau = 0.1$.

$p(u) - p_0(u)$

T_4
T_5
T_6

u

Figure 12: Deviations of $p(u)$ from the Gaussian $p_0(u)$ for the 4th to 6th Tchebyscheff-polynomial; $\tau = 0.1$.

$\frac{p(u)-p_0(u)}{\tau}$

T_4
T_5
analytic

u

Figure 13: Comparison of the analytic result (43) with the numerical result for T_4 and T_5; $\tau = 0.1$.

$\frac{p(u)-p_0(u)}{\tau}$

T_3
analytic

u

Figure 14: Comparison of the analytic result (44) with the numerical result for T_3; $\tau = 0.1$.

$\frac{p(u)-p_0(u)}{\sqrt{\tau}}$

T_2
analytic

u

Figure 15: Comparison of the analytic result (45) with the numerical result for T_2, $\tau = 0.06$.