Non-extensive statistical mechanics and particle spectra in elementary interactions

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Abstract

We generalize Hagedorn's statistical theory of momentum spectra of particles produced in high-energy collisions using Tsallis' formalism of non-extensive statistical mechanics. Suitable non-extensive grand canonical partition functions are introduced for both fermions and bosons. Average occupation numbers and moments of transverse momenta are evaluated in an analytic way. We analyse the energy dependence of the non-extensitivity parameter q as well as the q-dependence of the Hagedorn temperature. We also take into account the multiplicity. As a final result we obtain formulas for differential cross sections that are in very good agreement with e^+e^- annihilation experiments.

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1 Introduction

35 years ago Hagedorn has written a paper of fundamental importance [1] which nowadays is very often cited in various contexts of statistical mechanics, particle physics, and string theory. In this paper he developed a statistical description of momentum spectra of particles produced in collider experiments. At a fundamental level, of course, the underlying theory for this is quantum chromodynamics. However, for the hadronization cascade phenomenological models have to be used in addition to QCD. The complexity inherent in this process is so immense that usually one has to perform Monte Carlo simulations in order to explain the experimental measurements of cross sections.

Hagedorn's successful approach was to see this problem from a thermodynamic point of view (of course QCD was not known at the time he wrote the paper). He devoped a theory that is nowadays regularly in use do describe e.g. heavy ion collisions [2, 3, 4]. The basic assumption is that in the scattering region the density of states grows so rapidly that the temperature cannot exceed a certain maximum temperature, the Hagedorn temperature T_0. This is similar to a first oder phase transition of e.g. boiling water, just that the Hagedorn temperature describes 'boiling' nuclear matter at about $T_0 \approx 200$ MeV. This state of matter is closely related to what is nowadays called quark gluon plasma. For some theoretical approaches to transverse momentum spectra, see e.g. [5, 6, 7]. As already mentioned, the Hagedorn phase transition is also of fundamental interest in string theories [8, 9, 10].

While for collider experiments (e.g. e^+e^- annihilation) the Hagedorn theory yields a good description for relatively small center of mass enegies ($E < 10$ GeV), it fails at large energies. Indeed, Hagedorn's approach predicts an exponential decay of differential cross sections for large transverse momenta, whereas in experiments one observes non-exponential behaviour for large energies ($E > 10$ GeV) [11, 12, 13]. For this reason one usually restricts the comparison of the experimental results at higher energies to Monte Carlo simulations, which reasonably well reproduce the experimental data. But apparently, for a deeper understanding from a statistical mechanics point of view there is the need to generalize Hagedorn's original ideas.

In this paper we show that it is possible to extend Hagedorn's theory in such a way that it correctly describes the experimental findings at large energies as well. The basic input for this is the recently developed formalism of non-extensive statistical mechanics [14, 15, 16], a generalization of ordinary statistical mechanics suitable for multifractal and self-similar systems with long-range interactions. The formalism has been introduced by Tsallis 12 years ago [14] and has in the mean time been shown to be very successful to describe e.g. systems exhibiting turbulent behaviour [17, 18, 19] or Hamiltonian systems with long-range interactions [20]. Several other physical applications were described in [21]—[25]. Very recently, evidence was provided that the new formalism has applications in particle physics as well. Walton and Rafelski [26] studied a Fokker Planck equation describing charmed quarks in a thermal quark-gluon plasma and showed that Tsallis statistics is relevant. Wilk and Wlodarczyk [27] provided evidence that the distribution of depths of vertices of hadrons originating from cosmic ray cascades, as measured in emulsion chamber experiments, follows Tsallis statistics. Alberico, Lavagno and Quarati [28] successfully analysed heavy ion collisions using Tsallis statistics. Bediaga, Curado and Miranda [29] showed that differential cross sections in e^+e^- annihilation experiments can be very very well fitted using Tsallis statistics. In the present paper we will work out the theory underlying this.

To effectively describe the complex QCD interactions and hadronization cascade by a a thermodynamic model, it seems reasonable to use the Tsallis formalism (which contains ordinary statistical mechanics as a special case), since this formalism is specially designed to include selfsimilar systems and systems with long-range interactions. What happens in a collider experiment is expected to fall into this category, since QCD forces are strong at large distances. Moreover, the self-similarity of the scattering process was already recognized by Hagedorn, who described the various possible particle states as a 'fireball' and who defined a fireball as follows [30]: *A fireball is*

**... a statistical equilibrium of an undetermined number of all kinds of fireballs, each of which in turn is considered to be...*
(back to *)

Clearly, nowadays we would call this a self-similarity assumption.

In this paper we will combine Hagedorn's and Tsallis' approach. We will introduce a suitable generalized grand canonical partition function. The predictions following from this generalized thermodynamic model can be evaluated analytically (at least in a certain approximation). We will also take into account the multiplicity and suggest a concrete dependence of the non-extensitivity parameter q on the energy E. The result will be a concrete formula for differential cross sections of transverse momenta p_T where no free fitting parameters are left. Our formula turns out to be in excellent agreement with experimental measurements in e^+e^- annihilation experiments. This turns out to be true for the entire range of center of mass energies that have been probed in experiments. Our analytical formulas are also in good agreement with the curves obtained from Monte Carlo simulations, thus suggesting that the complexity inherent in these simulations is effectively reproduced by the simple thermodynamic model considered here.

This paper is organized as follows. In section 2 we shortly review the main results of Hagedorn's theory. In section 3 we shortly review Tsallis' theory. In section 4 we combine both. We will write down the generalized grand canonical partition function, including both fermions and bosons. In section 5 we will consider a large p_T approximation, which allows for a simple analytical treatment. We will numerically show that this approximation is a good one even for relatively small values of the transverse momentum. We derive the relevant form of the probability density and calculate all moments of transverse momenta. In section 6 we investigate the (weak) dependence of the Hagedorn temperature T_0 on the non-extensitivity parameter q. In section 7 we analyse the energy dependence of the parameter q. In section 8 we take into account the multiplicity and derive the final formula for the differential cross section. Finally, in section 9 we compare our theoretical results with the experimentally measured results of the TASSO and DELPHI collaboration. Our concluding remarks are given in section 10.

2 Hagedorn's theory

Hagedorn's theory [1] essentially predicts that the differential cross section in a scattering experiment is given by

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} = cp_T \int_0^\infty dx\; e^{-\beta\sqrt{x^2+\mu^2}} \tag{1}$$

Here p_T is the transverse momentum, $\mu = \sqrt{p_T^2+m^2}$ is the transverse mass, the integration variable x stands for the longitudinal momentum, $\beta = 1/(kT_0)$ denotes the inverse Hagedorn temperature, and c is some constant. The factor $e^{-\beta\sqrt{x^2+\mu^2}}$ is immediately recognized as a Boltzmann factor with energy given by the relativistic energy-momentum relation. In the following, we will set the Boltzmann constant k equal to 1.

The Hagedorn temperature T_0 is independent of the center of mass energy E of the beam and typically has a a value of about 100-300 MeV. The physical idea underlying Hagedorn's approach is that T_0 describes a kind of 'boiling temperature' of nuclear matter, which cannot be further increased by external energy transfer. Any further increase of energy just produces new states of particles rather than an increase of temperature.

The integral in eq. (1) can be further evaluated to yield

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} = cp_T\mu K_1(\beta\mu), \tag{2}$$

where K_1 is the modified Hankel function. For p_T large compared to both T_0 and m one obtains from the asymptotic behaviour of the Hankel function the approximate formula

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} \sim p_T^{3/2} e^{-\beta p_T}. \tag{3}$$

One sees that the differential cross section decays exponentially for large values of p_T. While this exponential decay is indeed observed for collider experiments with relatively small center of mass energies ($E < 10$ GeV), clear deviations from exponential decay have been observed at higher energies [11, 12, 13]. Here the Hagedorn theory is not valid any more and the dependence of the differential cross section on the transverse momentum is more complicated. Indeed, for large p_T polynomial decay is observed. For example, the ZEUS colloboration [13] has fitted their measurements obtained in ep collision experiments by an empirical power law of the form $(1 + const \cdot p_T)^{-\alpha}$, with exponent α measured as $\alpha = 5.8 \pm 0.5$.

In the following sections we will show that it is straightforward to extend the Hagedorn theory in such a way that it decribes experiments at high energies ($E > 10$ GeV) as well. The basic tool for this is the recently developed formalism of non-extensive statistical mechanics. It will just lead to asymptotic power laws of the above form.

3 Tsallis' theory

The formalism of non-extensive statistical mechanics is a generalization of the ordinary formalism of statistical mechanics [14]–[16]. Wheras ordinary statistical mechanics is derived by extremizing the Boltzmann-Gibbs entropy

$S = -\sum_i p_i \ln p_i$ (subject to constraints), in non-extensive statistical mechanics the more general Tsallis entropies

$$S_q = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right) \tag{4}$$

are extremized. The p_i are probabilities associated with the microstates of a physical system, and q is the non-extensivity parameter. The ordinary Boltzmann-Gibbs entropy is obtained in the limit $q \to 1$.

The Tsallis entropies are closely related to the Rényi information measures [31] and have similarly nice properties as the Boltzmann-Gibbs entropy has. They are positive, concave, take on their extremum for the uniform distribution, and preserve the Legendre transform structure of thermodynamics. However, they are not additive for independent subsystems (hence the name 'non-extensive' statistical mechanics). In the mean time a lot of physical applications have been reported for the formalism with $q \neq 1$. Examples are 3-dimensional fully developed hydrodynamic turbulence [17, 18, 19], 2-dimensional turbulence in pure electron plasmas [23, 24], Hamiltonian systems with long-range interactions [20], granular systems [21], systems with strange non-chaotic attractors [22], and peculiar velocities in Sc galaxies [25].

Given some set of probabilities p_i one can proceed to another set of probabilities P_i defined by

$$P_i = \frac{p_i^q}{\sum_i p_i^q}. \tag{5}$$

These distributions are called escort distributions [32]. Extremizing S_q under the energy constraint

$$\sum_i P_i \epsilon_i = U_q, \tag{6}$$

where the ϵ_i are the energy levels of the microstates, the probabilities P_i come out of the extremization procedure as

$$P_i = \frac{1}{Z_q}(1 + (q-1)\beta\epsilon_i)^{-\frac{q}{q-1}}, \tag{7}$$

where

$$Z_q = \sum_i (1 + (q-1)\beta\epsilon_i)^{-\frac{q}{q-1}} \tag{8}$$

is the partition function and $\beta = 1/T$ is a suitable inverse temperature variable (depending on U_q). In the limit $q \to 1$, ordinary statistical mechanics is recovered, and the P_i just reduce to the ordinary canonical distributions $P_i = \frac{1}{Z} e^{-\beta \epsilon_i}$. For $q \neq 1$, on the other hand, they can be regarded as generalized versions of the canonical ensemble, describing probability distributions in a complex non-extensive system at inverse temperature β.

4 Combining Hagedorn's and Tsallis' theory

4.1 States of factorizing probabilities

To generalize Hagedorn's theory we first have to decide on how to introduce grand canonical partition functions in non-extensive statistical mechanics. The problem is non-trivial, as one immediately recognizes from just considering a system of N independent particles. In fact, it must first be defined what one means by independence in the non-extensive approach. The most plausible definition is that for independent particles probabilities should factorize. Let us consider such a distinguished factorized state where each joint probability is given by products of single-particle Tsallis distributions. This means, the joint probability to $P_{i_1,i_2,\ldots,i_N}$ to observe particle 1 in energy state ϵ_{i_1}, particle 2 in energy state ϵ_{i_2}, and so on is given by

$$P_{i_1,i_2,\ldots,i_N} = \frac{1}{Z} \prod_{j=1}^{N} (1 + (q-1)\beta\epsilon_{i_j})^{-\frac{q}{q-1}}. \tag{9}$$

At the same time, we could also describe our system by the Tsallis distribution formed with the total energy (the Hamiltonian $H(i_1, i_2, \ldots, i_N)$ of the system)

$$P_{i_1,i_2,\ldots,i_N} = \frac{1}{Z}(1 + (q-1)\beta H(i_1, \ldots, i_N))^{-\frac{q}{q-1}}. \tag{10}$$

Equating (9) and (10) we see that the total energy of the system is given by

$$1 + (q-1)\beta H = \prod_{j=1}^{N} (1 + (q-1)\beta\epsilon_{i_j}), \tag{11}$$

which we may write as

$$H = \sum_j \epsilon_{i_j} + (q-1)\beta \sum_{j,k} \epsilon_{i_j}\epsilon_{i_k} + (q-1)^2\beta^2 \sum_{j,k,l} \epsilon_{i_j}\epsilon_{i_k}\epsilon_{i_l} + \cdots \tag{12}$$

(all summation indices are pairwise different). This means that for the unique particle state where probabilities factorize the total energy of the system is not the sum of the single particle energies, provided $q \neq 1$. In other words, if seen from the energy point of view, the particles are formally interacting with a coupling constant $(q-1)\beta$ although the probabilities factorize. This fact is not too surprising— the formalism of non-extensive statistical mechanics is of course designed to describe systems with (long-range) interactions, and also the entropy is non-additive.

We can also invert the above statement. If we consider a Hamiltonian that is just the sum of the single particle energies, then the probabilities do not factorize provided $q \neq 1$. This is a well-known statement in non-extensive statistical mechanics.

In the following, we will generalize Hagedorn's theory using the unique states of factorizing probabilities—thus implicitly introducing interactions between the particles from the energy point of view.

4.2 Statistical mechanics of the fireball

We will consider particles of different types and label the particle types by an index j. Each particle can be in a certain momentum state labelled by the index i. The energy associated with this state is

$$\epsilon_{ij} = \sqrt{\mathbf{p}_i^2 + m_j^2} \tag{13}$$

(the relativistic energy-momentum relation). Let us define a non-extensive Boltzmann factor by

$$x_{ij} = (1 + (q-1)\beta\epsilon_{ij})^{-\frac{q}{q-1}}. \tag{14}$$

It approaches the ordinary Boltzmann factor $e^{-\beta\epsilon_{ij}}$ for $q \to 1$. We now very much follow Hagedorn's original ideas, replacing the ordinary Boltzmann factor by the generalized one. The generalized grand canonical partition function is introduced as

$$Z = \sum_{(\nu)} \prod_{ij} x_{ij}^{\nu_{ij}} \tag{15}$$

Here ν_{ij} denotes the number of particles of type j in momentum state i. The sum $\sum_{(\nu)}$ stands for a summation over all possible particle numbers. For bosons one has $\nu_{ij} = 0, 1, 2, \ldots, \infty$, whereas for fermions one has $\nu_{ij} = 0, 1$. It follows that for bosons

$$\sum_{\nu_{ij}} x_{ij}^{\nu_{ij}} = \frac{1}{1 - x_{ij}} \qquad (bosons) \tag{16}$$

whereas for fermions

$$\sum_{\nu_{ij}} x_{ij}^{\nu_{ij}} = 1 + x_{ij} \qquad (fermions) \tag{17}$$

Hence the partition function can be written as

$$Z = \prod_{ij} \frac{1}{1 - x_{ij}} \prod_{i'j'} (1 + x_{i'j'}), \tag{18}$$

where the prime labels fermionic particles. For the logarithm of the partition function we obtain

$$\log Z = -\sum_{ij} \log(1 - x_{ij}) + \sum_{i'j'} \log(1 + x_{i'j'}). \tag{19}$$

One may actually proceed to continuous variables by replacing

$$\sum_i [...] \rightarrow \int_0^\infty \frac{V_0 4\pi p^2}{h^3} [...] dp = \frac{V_0}{2\pi^2} \int_0^\infty p^2 [...] dp \quad (\hbar = 1) \tag{20}$$

(V_0: volume of the interaction region) and

$$\sum_j [...] \rightarrow \int_0^\infty \rho(m) [...] dm, \tag{21}$$

where $\rho(m)$ is the mass spectrum.

Let us now calculate the average occupation number of a particle of species j in the momentum state i. We obtain

$$\begin{aligned} \bar{\nu_{ij}} = x_{ij} \frac{\partial}{\partial x_{ij}} \log Z &= \frac{x_{ij}}{1 \pm x_{ij}} \\ &= \frac{1}{(1 + (q-1)\beta\epsilon_{ij})^{\frac{q}{q-1}} \pm 1} \end{aligned} \tag{22}$$

where the $-$ sign is for bosons and the $+$ sign for fermions.

In order to single out a particular particle of mass m_0, one can formally work with the mass spectrum $\rho(m) = \delta(m - m_0)$. To obain the probability to observe a particle of mass m_0 in a certain momentum state, we have to multiply the average occupation number with the available volume in momentum space. An infinitesimal volume in momentum space can be written as

$$dp_x dp_y dp_z = dp_L p_T \sin\theta dp_T d\theta \tag{23}$$

where $p_T = \sqrt{p_y^2 + p_z^2}$ is the transverse monentum and $p_x = p_L$ is the longitutinal one. By integrating over all θ and p_L one finally arrives at a probability density $w(p_T)$ of transverse momenta given by

$$w(p_T) = const \cdot p_T \int_0^\infty dp_L \frac{1}{(1 + (q-1)\beta\sqrt{p_T^2 + p_L^2 + m_0^2})^{\frac{q}{q-1}} \pm 1}. \tag{24}$$

Since the Hagedorn temperature is rather small (of the order of the π mass), under normal circumstances one has $\beta\sqrt{p_T^2 + p_L^2 + m_0^2} >> 1$, and hence the ± 1 can be neglected if q is close to 1. One thus obtains for both fermions and bosons the statistics

$$w(p_T) \approx const \cdot p_T \int_0^\infty dp_L (1 + (q-1)\beta\sqrt{p_T^2 + p_L^2 + m_0^2})^{-\frac{q}{q-1}} \tag{25}$$

which, if our theory is correct, should determine the p_T dependence of experimentally measured particle spectra. The differential cross section $\sigma^{-1} d\sigma/dp_T$ is expected to be proportional to $w(p_T)$.

5 Large p_T approximation

5.1 The differential cross section

To further proceed with analytic calculations, one may actually perform one further approximation step. This is the generalization of the step of going from eq. (1) to eq. (3) in Hagedorn's original theory, which is a good approximation for large p_T. Since for our applications q is close to 1, one expects that a similar step is possible in the more general non-extensive theory.

Let us write the formula for the differential cross section in the form

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} = cu \int_0^\infty dx \, \left(1 + (q-1)\sqrt{x^2 + u^2 + m_\beta^2}\right)^{-\frac{q}{q-1}}. \tag{26}$$

Here $x = p_L/T_0$, $u = p_T/T_0$ and $m_\beta := m_0/T_0$ are the longitudinal momentum, transverse momentum and mass in units of the Hagedorn temperature T_0, respectively. c is a suitable constant. Let us look at this formula for large values of p_T. If u is very large, we can approximate $\sqrt{x^2 + u^2 + m_\beta^2} = u\sqrt{1 + (x^2 + m_\beta^2)/u^2} \approx u + (x^2 + m_\beta^2)/(2u)$. Of course, for this to be true the integration variable x should not be too large, but for large x the integrand is small anyway and yields only a very small contribution to the cross section. Moreover, since u is large we may also neglect the mass term m_β^2, arriving at the aprroximation

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} \approx cu \int_0^\infty dx \left(1 + (q-1)\left(u + \frac{x^2}{2u}\right)\right)^{-\frac{q}{q-1}} \tag{27}$$

Although this may look like a rather crude approximation, in practice it is quite a good one. This is illustrated in Fig. 1, which shows the right hand sides of eq. (26) and eq. (27) for $q = 1, 1.1, 1.2$. The lines corresponding to the exact expression (26) and the approximation (27) can hardly be distinguished. The range of u and q is similar to what we will use later for the comparison with experimental measurements. Since the Hagedorn temperature T_0 is about 120 MeV, and since very often pions are produced, we have chosen in formula (26) $m_\beta^2 = (m_\pi/T_0)^2 \approx 1.3$. The errors made in eq. (27) by neglecting m_β and by approximating the square root work in opposite directions and almost cancel each other for $q \approx 1.1$.

Within this approximation we can now easily proceed by analytical means. We may write eq. (27) as

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} = cu\,(1 + (q-1)u)^{-\frac{q}{q-1}} \int_0^\infty dx \left(1 + \frac{q-1}{2u(1+(q-1)u)}x^2\right)^{-\frac{q}{q-1}} \tag{28}$$

Substituting

$$t := \sqrt{\frac{q-1}{2u(1+(q-1)u)}}\; x \tag{29}$$

and using the general formula [33]

$$\int_0^\infty \frac{t^{2x-1}}{(1+t^2)^{x+y}} dt = \frac{1}{2}B(x,y) \qquad (Re\ x > 0, Re\ y > 0), \tag{30}$$

where

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{31}$$

denotes the beta-function, one arrives at

$$\frac{1}{\sigma}\frac{d\sigma}{dp_T} = c(2(q-1))^{-1/2} B\left(\frac{1}{2}, \frac{q}{q-1} - \frac{1}{2}\right) u^{3/2}\,(1 + (q-1)u)^{-\frac{q}{q-1}+\frac{1}{2}}. \tag{32}$$

This formula, with suitably determined c, q, T_0, will turn out to be in very good agreement with experimentally measured cross sections.

5.2 Normalization

Let us quite generally consider a probability density with the above u-dependence

$$p(u) = \frac{1}{Z_q}\, u^{3/2}\, (1+(q-1)u)^{-\frac{q}{q-1}+\frac{1}{2}} \tag{33}$$

Normalization yields

$$Z_q = \int_0^\infty u^{3/2}\, (1+(q-1)u)^{-\frac{q}{q-1}+\frac{1}{2}}\, du. \tag{34}$$

Substituting $t^2 := (q-1)u$ the integral can be brought into the form (30), and one obtains

$$Z_q = (q-1)^{-5/2} B\left(\frac{5}{2}, \frac{q}{q-1} - 3\right) \tag{35}$$

Note that the beta function further simplifies if $\frac{q}{q-1} - 3$ is an integer. For $n \in \mathbf{N}$ one generally has

$$B(x,n) = \frac{(n-1)!}{x(x+1)(x+2)\cdots(x+n-1)}, \tag{36}$$

in our case $x = \frac{5}{2}$.

5.3 Moments

Let us now evaluate the moments of u defined by

$$\langle u^m \rangle = \int_0^\infty u^m p(u) du = \frac{1}{Z_q} \int_0^\infty u^{\frac{3}{2}+m}\, (1+(q-1)u)^{-\frac{q}{q-1}+\frac{1}{2}}\, du. \tag{37}$$

Again substituting $t^2 = (q-1)u$ the integral can be evaluated to give

$$\begin{aligned} \langle u^m \rangle &= \frac{1}{(q-1)^m} \frac{B\left(\frac{5}{2}+m, \frac{q}{q-1} - 3 - m\right)}{B\left(\frac{5}{2}, \frac{q}{q-1} - 3\right)} \\ &= \frac{1}{(q-1)^m} \frac{\Gamma\left(\frac{5}{2}+m\right)\Gamma\left(\frac{q}{q-1} - 3 - m\right)}{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{q}{q-1} - 3\right)} \end{aligned} \tag{38}$$

Generally the Gamma function satisfies

$$\Gamma(x+1) = x\Gamma(x) \tag{39}$$

which one may iterate to obtain

$$\Gamma(x+m) = \Gamma(x) \prod_{j=0}^{m-1} (x+j). \tag{40}$$

Using this in eq. (38) one finally arrives at

$$\langle u^m \rangle = \frac{1}{2^m} \prod_{j=0}^{m-1} \frac{5+2j}{4+j-(3+j)q}. \tag{41}$$

In particular, one obtains for the average of u

$$\langle u \rangle = \frac{1}{2} \frac{5}{4-3q} \tag{42}$$

and for the second and third moment

$$\langle u^2 \rangle = \frac{1}{4} \frac{5}{4-3q} \frac{7}{5-4q} \tag{43}$$

$$\langle u^3 \rangle = \frac{1}{8} \frac{5}{4-3q} \frac{7}{5-4q} \frac{9}{6-5q}. \tag{44}$$

The variance is given by

$$\sigma^2 := \langle u^2 \rangle - \langle u \rangle^2 = \frac{5}{4} \frac{3-q}{(4-3q)^2(5-4q)}. \tag{45}$$

Note that the moments obey the simple recurrence relation

$$\langle u^{m+1} \rangle = \langle u^m \rangle \cdot \frac{5+2m}{4+m-(3+m)q}. \tag{46}$$

6 q-dependence of the Hagedorn temperature

A fundamental property of Hagedorn's theory is the fact that the Hagedorn temperature T_0 is independent of the center of mass energy E. Now, in the generalized theory we have a new parameter q and it is a priori not clear if and how T_0 depends on q.

However, looking at eq. (12) we recognize that if q increases from 1 to slightly larger values the (formal) interaction energy of our system increases.

This increase in interaction energy must come from somewhere and is expected to be taken from the (finite volume) heat bath of the fireball. Thus the effective temperature of the bath is expected to slightly decrease with increasing q.

To get a rough estimate of this effect, let us work within the approximation scheme of the previous section. Differentiating eq. (33) with respect to u one immediately sees that the distribition $p(u)$ has a maximum at

$$u^* = \frac{p_T^*}{T_0} = \frac{3}{3-q}. \tag{47}$$

On the other hand, the experimentally measured cross sections always appear to have their maximum at roughly the same value of the transverse momentum p_T, namely at

$$p_T^* \approx 180\ MeV, \tag{48}$$

independent of the beam energy E. This implies that in the generalized thermodynamic approach the effective Hagedorn temperature T_0 will become (slightly) q-dependent.

$$T_0 = \left(1 - \frac{q}{3}\right) p_T^* \tag{49}$$

The variation of T_0 with energy E is actually very weak. In the next section we will consider two extreme cases, namely $q \to 1$ for $E \to 0$ and $q \to \frac{11}{9}$ for $E \to \infty$. The first case corresponds to $T_0 = 120$ MeV, the second case to $T_0 = 107$ MeV. This is only a very small variation.

7 Energy dependence of q

We still have to decide how the non-extensitivity parameter q depends on the center of mass energy E of the beam. We will present some theoretical arguments, which are well supported by the experimental data.

7.1 Plausible value for q_{max}

Let us define α to be the power of the term $(1 + (q-1)u)^{-1}$ in eq. (33)

$$\alpha = \frac{q}{q-1} - \frac{1}{2}. \tag{50}$$

Clearly, for small energies E Hagedorn's theory ($q = 1$) is valid:

$$E \to 0 \Longrightarrow \alpha \to \infty \Longleftrightarrow q \to 1 \tag{51}$$

For increasing E, α should decrease. For example, the ZEUS collaboration measures $\alpha = 5.8 \pm 0.5$ at medium energies [13]. However, α cannot become arbitrarily small, because we must have a finite variance of u (otherwise statistical mechanics does not make sense). For large u, we can certainly use the large p_T approximation of section 5. Asymptotically the density $p(u)$ decays as

$$p(u) \sim u^{-\alpha+3/2} \tag{52}$$

Thus $u^2 p(u)$ decays as $u^{-\alpha+7/2}$ and hence $\langle u^2 \rangle = \int du\, u^2 p(u)$ only exists if

$$-\alpha + \frac{7}{2} < -1 \Longleftrightarrow \alpha > \frac{9}{2} \Longleftrightarrow q < \frac{5}{4} \tag{53}$$

(for $\alpha = 9/2$ there is a logarithmic divergence). This is a strict upper bound on q.

If, in addition, we postulate that the distinguished limit theory corresponding to the largest possible value of q should be analytic in $\sqrt{u}$, then only integer and half-integer values of α are allowed. This leads to

$$\alpha_{min} = 5 \Longleftrightarrow q_{max} = \frac{11}{9} = 1.222 \tag{54}$$

as the smallest possible value of α, respectively largest value of q.

7.2 A smooth dependence on q

It is reasonable to assume (and supported by the experimental results) that the smallest possible value of the exponent α is taken on for the largest possible energy E. Hence we have the two limit cases

$$E \to 0 \quad \Longrightarrow \quad \alpha \to \infty \tag{55}$$

$$E \to \infty \quad \Longrightarrow \quad \alpha \to \alpha_{min} = 5. \tag{56}$$

A smooth monotonously decreasing interpolation between these limit cases is given by the formula

$$\alpha(E) = \frac{5}{1 - e^{-E/E_0}}, \tag{57}$$

where E_0 is some suitable energy scale. It turns out that the experimental data are perfectly fitted if we choose

$$E_0 = \frac{1}{2} m_Z = 45.6 \; GeV, \tag{58}$$

m_Z being the mass of the Z^0 boson.

Whether this coincidence of the relaxation parameter E_0 with half of the Z^0 mass is a random effect or whether there is physical meaning behind this is still an open question. What is clear is the following. At center of mass energies of the order 90 GeV often a massive elektroweak gauge boson Z^0 is produced. If a Z^0 at rest decays into a quark pair $q\bar{q}$, then each quark has the initial energy $\frac{1}{2} m_Z$, which is input to the hadronization cascade. It is obvious that at this energy scale there is a rapid change in the behaviour of the fireball due to the occurence of the electroweak bosons. The same mass scale now seems to mark the crossover scale between ordinary Hagedorn thermostatistics, valid for $E << \frac{1}{2} m_Z$, and generalized thermostatistics with $q \approx q_{max}$, valid for $E >> \frac{1}{2} m_Z$.

Eq. (57) can equivalently be written as

$$q(E) = \frac{11 - e^{-E/E_0}}{9 + e^{-E/E_0}}. \tag{59}$$

8 Energy dependence of the multiplicity

The differential cross section $\sigma^{-1} d\sigma / dp_T$ measured in experiments is not a normalized probability density. This is due to the fact that it is dependent on the multiplicity M (the average number of produced charged particles). Also, it has dimension GeV^{-1}, whereas the probability density $p(u)$ is dimensionless. All this, however, is just a question of normalization.

To proceed from $p(u)$ to $\sigma^{-1} d\sigma / dp_T$, we should multiply $p(u)$ with the multiplicity M. Also, in order to give the correct dimension of a cross section, we should multiply with T_0^{-1}. Thus we arrive at the formula

$$\frac{1}{\sigma} \frac{d\sigma}{dp_T} = \frac{1}{T_0} M p(u). \tag{60}$$

The multiplicity M as a function of the beam energy E has been independently measured in many experiments. A good fit of the experimental data in the relevant energy region is the formula

$$M = \left(\frac{E}{T_0^{q=1}}\right)^{5/11} \qquad T_0^{q=1} = 120\ MeV \tag{61}$$

(see Fig. 2, data as collected in [34]). It is remarkable that the constant in front of this power law is 1 if E is measured in units of the Hagedorn Temperature $T_0^{q=1}$. Apparently, the Hagedorn temperature is a very appropriate portion of energy for our statistical approach. Essentially, the scaling law (61) says that only a certain fraction of the logarithm of the energy E (described by the scaling exponent $5/11 \approx 0.45$) is used to increase the number of charged particles.

9 Comparison with experimentally measured cross sections

We are now in a position to directly compare with experimental measurements of cross sections. All parameters of the generalized Hagedorn theory (the Hagedorn temperature $T_0(q)$, the non-extensitivity parameter $q(E)$ and the normalization constant of the cross section) have been discussed in the previous sections and concrete equations have been derived.

Formula (60) with $p(u)$ given by eq.(33), $q(E)$ given by eq. (59), $T_0(q)$ given by eq.(49) and multiplicity $M(E)$ given by eq. (61) turns out to very well reproduce the experimental results of cross sections for all energies E. This is illustrated in Fig. 3, which shows experimental cross sections measured by the TASSO and DELPHI collaborations ([11, 12], data as collected in [34]) as well as our theoretical prediction. For all curves we have chosen the same universal parameters $p_T^* = 180$ MeV and $E_0 = \frac{1}{2}m_Z$, so we do not use any energy-dependent fitting parameters.

The agreement is remarkably good. Hence the statistical approach presented in this paper qualifies as a simple thermodynamic model that explains the experimental data quite well. In particular, for the first time analytical formulas are obtained that correctly describe the measured cross sections. The quality of agreement is at least as good as that of Monte Carlo simulations, in spite of the fact that Monte Carlo simulations usually use a large number of free parameters. For small p_T, the agreement seems even to be slightly better than that of Monte Carlo simulations (see e.g. [34] for typical Monte Carlo results).

Our approach yields q-values of similar order of magnitude as the ones used in the fits in [29]. In our theoretical approach all parameters are now given by concrete formulas. Some of these formulas, most importantly eq. (59), have been derived empirically and should thus be further checked and possibly refined by further experimental measurements.

10 Conclusion

In this paper we have developed a thermodynamic model describing the statistics of transverse momenta of particles produced in high-energy collisions. Our approach is based on a generalization of Hagedorn's theory using Tsallis' formalism of non-extensive statistical mechanics. Hagedorn's original theory is recovered for small center of mass energies, whereas for larger energies deviations from ordinary Boltzmann-Gibbs statistics become relevant. The crossover energy scale between the two regimes $E \to 0$, $q \approx 1$ and $E \to \infty$, $q \approx \frac{11}{9}$ is given by half of the Z^0 mass.

At large energies the generalized thermodynamic theory implies that cross sections decay with a power law. This power law is indeed observed in experiments. Our theory allows us to analytically evaluate moments of arbitrary order (within a certain approximation). We obtain formulas for differential cross sections that are in very good agreement with experimentally measured cross sections in annihilation experiments.

We suggest that future analysis of the experimental data should concentrate on precision measurements of the energy dependence of the parameter q. For this the mass spectrum of produced particles should carefully be taken into account. Generally, it appears that high-energy collider experiments do not only yield valuable information on particle physics, but they may also be regarded as ideal test grounds to verify new ideas from statistical mechanics.

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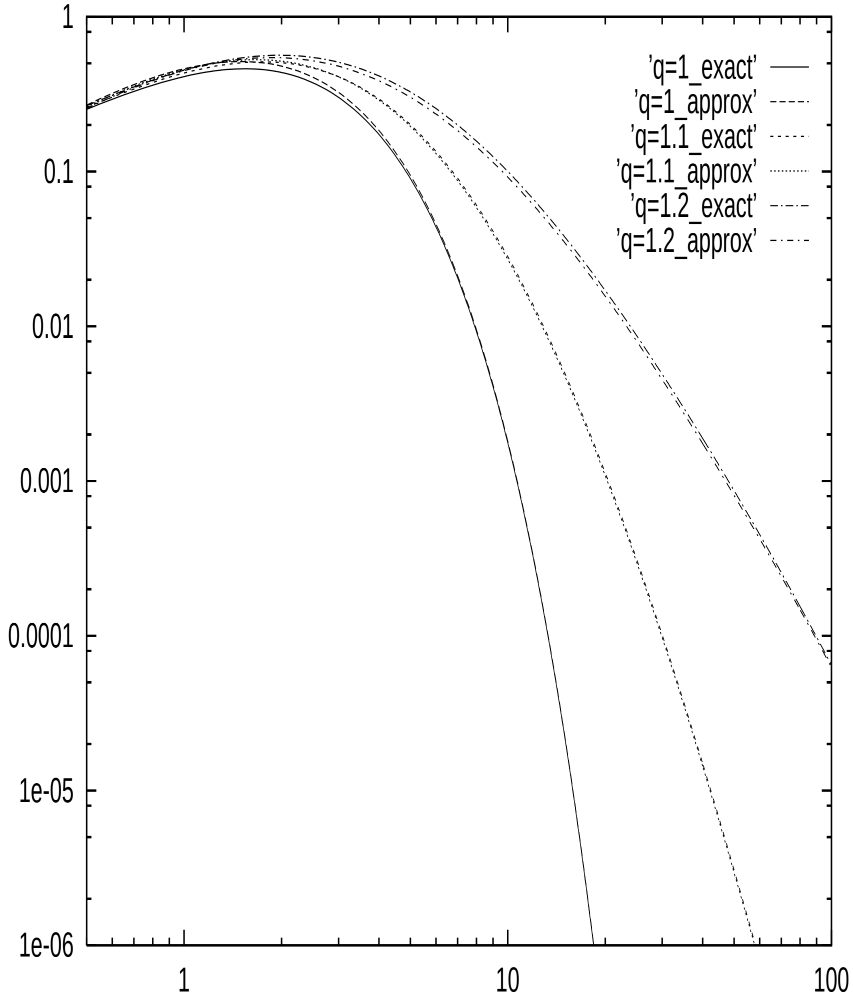
Figure captions

Fig. 1 Comparison between the exact formula (26) with $m_\beta^2 = (m_\pi/T_0)^2 = 1.3$ and the approximation (27) for various values of q. The constant c was set equal to 1 for all q.

Fig. 2 Dependence of the multiplicity M on the center of mass energy E. The figure shows the experimental data and a straight line that corresponds to the scaling law (61).

Fig. 3 Differential cross section as a function of the transverse momentum p_T for various center of mass energies E. The data correspond to measurements of the TASSO ($E \leq 44$ GeV) and DELPHI ($E \geq 91$ GeV) collaboration. The solid lines are given by the analytic formula (60).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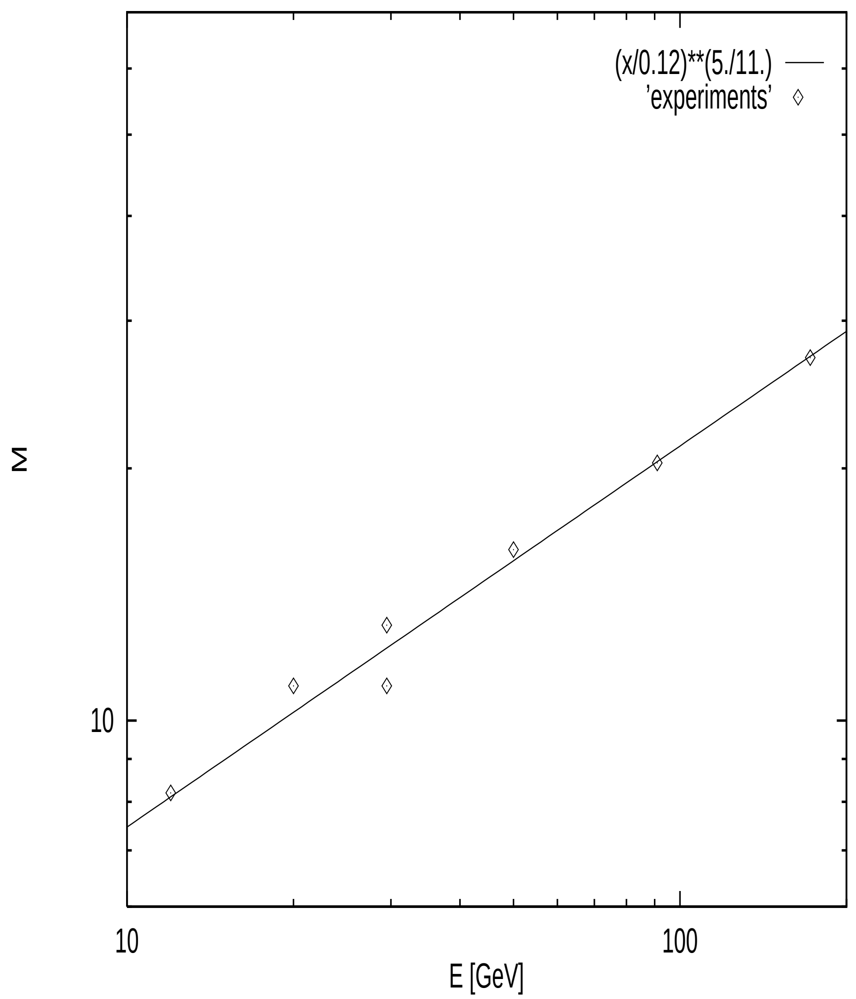
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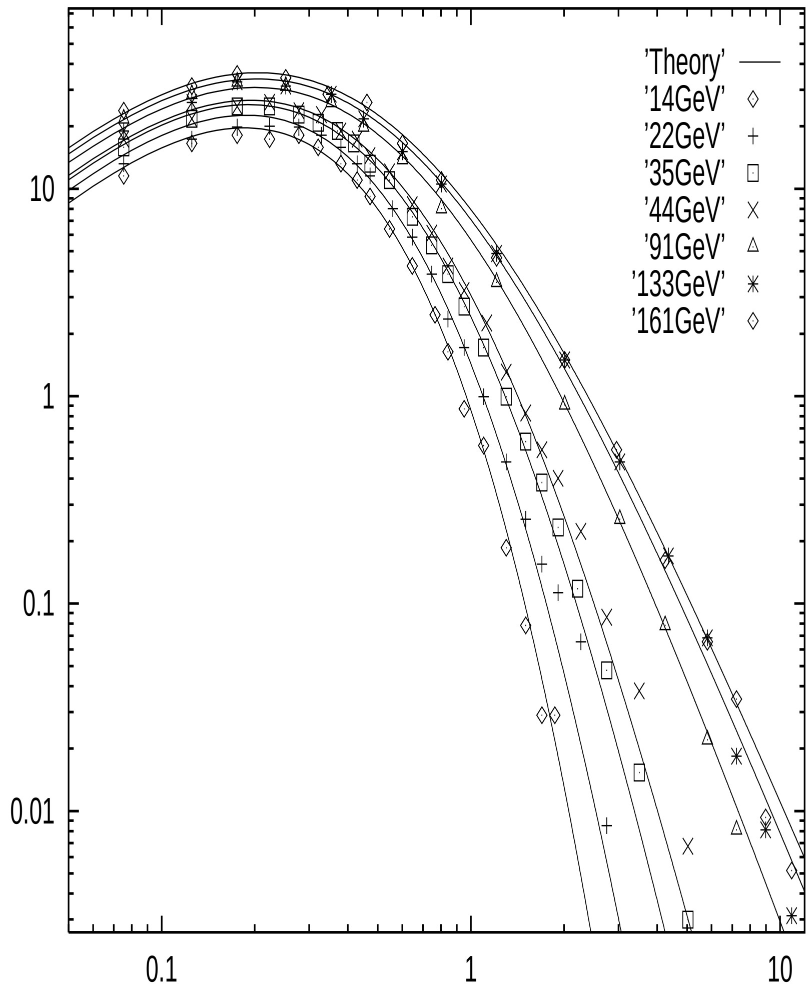
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