NL2602 Multifractal analysis

A multifractal is a fractal with a probability measure on its support. Typical examples are attractors of nonlinear mappings, where the relevant probability measure is given by the invariant measure of the map. Often quite a complicated structure is observed, and the invariant density may diverge at infinitely many points in the phase space with different exponents, i.e. an entire spectrum of singularites is generated.

To statistically analyze this complex behavior, it is useful to evaluate the so-called Rényi dimensions D_q (Rényi, 1970). These are generalizations of the usual box dimension (or capacity) that contain information not only on the topological structure of the fractal but on the probability measure as well. The simplest definition of the Rényi dimensions is as follows. Cover the fractal with small d-dimensional cubes ("boxes") of volume ϵ^d, where ϵ is the side length of the box and d is an integer dimension large enough to embed the fractal. For each such cube labeled by i we consider the probability p_i that is associated with it:

$$p_i = \int_{i-th\ box} \rho(\vec{x}) d\vec{x} \tag{1}$$

Here $\rho(\vec{x})$ is the probability density considered. The Rényi dimensions are defined for any real parameter q as

$$D_q = \lim_{\epsilon \to 0} \frac{1}{\log \epsilon} \frac{1}{q-1} \log \sum_i p_i^q, \tag{2}$$

where the sum is over all i with $p_i \neq 0$. There are also more sophisticated definitions of the Rényi dimensions based on boxes of variable size (analogous to the definition of the Hausdorff dimension), see e.g. Beck & Schlögl, 1993, for more details.

Generally, for a complicated multifractal (with lots of singularities of the density) the D_q depend on q, whereas for "simple" multifractals all D_q are the same and equal to the Hausdorff dimension. By changing the parameter q one "scans" the structure of the multifractal. Large q give more weight to large probabilities p_i, small q pronounce small probabilities p_i. Useful special cases of the Rényi dimensions are the box dimension (or capacity) D_0 (usually equal to the Hausdorff dimension, up to pathological cases), the information dimension D_1 (more precisely given by the limit $\lim_{q \to 1} D_q$) and the correlation dimension D_2. The correlation dimension can be easily extracted from experimental time series using the Grassberger-Procaccia algorithm (Grassberger & Procaccia, 1983). It is also of relevance for the estimation of typical period lengths that are generated due to computer roundoff errors in chaotic dynamical systems (Beck, 1989). Important are also the limit dimensions $D_{\pm}\infty$ obtained for $q \to \pm\infty$. They decribe the scaling behavior of the invariant density in the region of the phase space where the measure is most concentrated (D_∞), respectively least concentrated ($D_{-\infty}$).

From the Rényi dimensions one can proceed to the spectrum of singularities, or $f(\alpha)$ spectrum, by a Legendre transformation. The basic idea is very similar to thermodynamics. In fact, many of the ideas of the multifractal formalism are related to early work by Sinai, Ruelle and Bowen on the so-called "thermodynamic formalism" of dynamical systems (Ruelle, 1978). For multifractals one can regard the function $\tau(q) = (q-1)D_q$ as a kind of free energy, and q as a kind of inverse temperature (Tél, 1988; Beck & Schlögl, 1993). One then defines the variable α (the "internal energy") by $\partial\tau/\partial q =: \alpha$ and proceeds to the $f(\alpha)$ spectrum (the "entropy") by Legendre transformation:

$$f(\alpha) = q\alpha - \tau(q) \tag{3}$$

The advantage of the $f(\alpha)$ spectrum is that it has a kind of "physical meaning". Roughly speaking, it is the fractal dimension of the subset of points for which the probability density scales with a local exponent α, i.e. $p_i \sim \epsilon^{\alpha_i}$ with $\alpha_i \in [\alpha, \alpha + d\alpha]$. Hence this is a kind of statistical mechanics of local Hölder indices.

Let us consider a simple example of a multifractal. It is given by the classical cantor set with a multiplicative (non-uniform) measure (Fig. 1). Remember that the classical cantor set is constructed by cutting out the middle third out of the unit interval, then cutting out the middle third out of the remaining two intervals, and so on. In this way, at the k-the construction step there are 2^k intervals of length $(1/3)^k$. We now attribute a product measure to each of the intervals, according to the rule sketched in Fig. 1, with $w_1 + w_2 = 1$ but $w_1 \neq w_2$. In the limit $k \to \infty$ we obtain a multifractal. Let us calculate the corresponding Rényi dimensions. We cover the multifractal with little intervals of size $\epsilon_k = (1/3)^k$. The number of boxes with probability $p_i = w_1^j w_2^{k-j}$ is

$$N_{k,j} = \begin{pmatrix} j \\ k \end{pmatrix} = \frac{j!}{k!(k-j)!}. \tag{4}$$

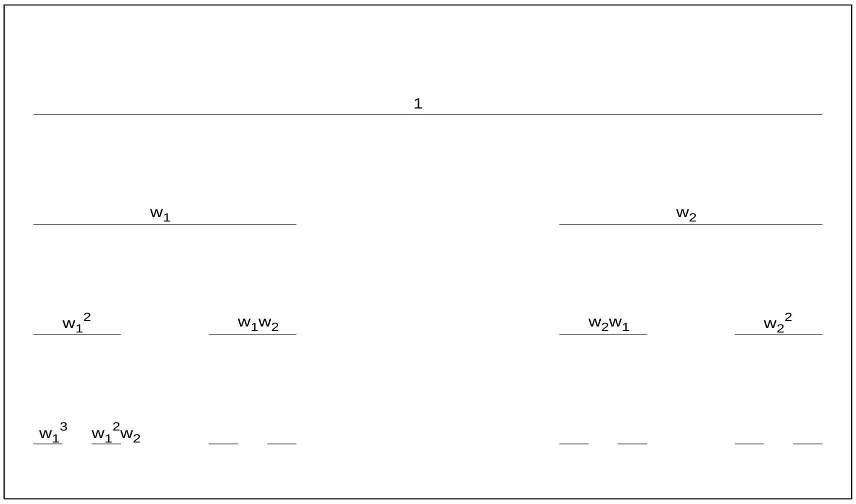

Figure 1: Construction of a classical cantor set with a multiplicative non-uniform measure.

Hence

$$\sum_i p_i^q = \sum_{j=0}^{k} N_{k,j} w_1^{jq} w_2^{(k-j)q} = (w_1^q + w_2^q)^k \tag{5}$$

and for the Rényi dimensions one obtains from the definition (2)

$$D_q = \frac{\log(w_1^q + w_2^q)}{(1-q)\log 3}. \tag{6}$$

In particular, the box dimension D_0 is given by the value $D_0 = \log 2/\log 3$, independent of w_1 and w_2. The other dimensions depend on the probabilities w_j, for example for the information dimension one obtains $D_1 = -(1/\log 3)\sum_{j=1}^{2} w_j \log w_j$ by considering the limit $q \to 1$. The Legendre transform of (6) yields the $f(\alpha)$ spectrum.

Another interesting example is the attractor of the logistic map at the critical point of period doubling accumulation. Fig. 2a shows the corresponding Rényi dimensions and Fig. 2b the corresponding $f(\alpha)$ spectrum. Generally, the value of α where the function $f(\alpha)$ has its maximum is equal to the Hausdorff dimension of the attractor. The value of $f(\alpha)$ where the function has slope 1 is equal to the information dimension.

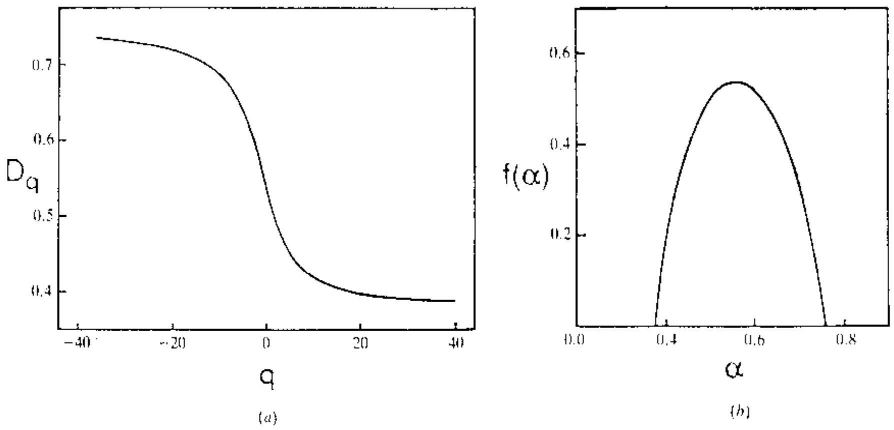

Figure 2: (a) Rényi dimensions and (b) $f(\alpha)$ spectrum of the Feigenbaum attractor.

In practice, one often wants to evaluate the Rényi dimensions (or the $f(\alpha)$ spectrum) for a given times series (Kantz & Schreiber, 1997) of experimental data without explicitly knowing the underlying dynamics or the invariant measure. Here various interesting methods are known (Grassberger-Procaccia algorithm (Grassberger & Procaccia, 1983), wavelet analysis (Arneodo et al., 1995), etc). These algorithms can be implemented without explicitly knowing the underlying dynamics. The wavelet transform of an experimental signal $s(x)$ is defined as

$$W_\Psi(x_0, a) = \frac{1}{a} \int_{-\infty}^{+\infty} \Psi^* \left(\frac{x - x_0}{a} \right) s(x) dx, \tag{7}$$

($*$: complex conjugate), where the analyzing wavelet Ψ is some localized function, often chosen to be the N-th derivative of a Gaussian function. For small a, the wavelet transform extracts local Hölder exponents from the signal s, and q-th moments of W_Ψ can then be used to define suitable partition functions which yield the Rényi dimensions.

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see also fractals, free energy, wavelets

Further reading:

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