Nonextensive statistical mechanics approach to fully developed hydrodynamic turbulence

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Abstract

We apply nonextensive methods to the statistical analysis of fully developed turbulent flows. Probability density functions (PDFs) of velocity differences obtained by extremization of the Tsallis entropies look very similar to what is measured in turbulence experiments. In a 0-th order perturbative approach the densities are symmetric, in 1st-oder there are universal skewness corrections due to higher-order correlations of the chaotic dynamics. We also consider the moment scaling exponents and show that the simplest model assumptions lead to scaling exponents similar to those of a K61 theory, but other approaches are possible as well. We derive a simple relation between the flatness factor F and the non-extensitivity parameter q, which can be used to directly extract the function $q(r)$ from experimentally measured structure functions.

1 Introduction

One may ask whether fully developed turbulent flows in 3 dimensions are distinguished in the sense that they extremize certain suitable information measures. It is clear that a fully developed turbulent state is a distinguished hydrodynamic state that has strongest possible stochastic properties subject to certain constraints. At the same time this stochastic process is very complicated and arises from a completely deterministic spatio-temporal chaotic evolution dynamics, which is highly nonlinear. The idea is to look for an extremization principle. This idea is actually not new. For example, Poppe [**?**] has presented some work on turbulence where he extremizes the Fisher entropy, and Castaing [] also starts from an extremum principle to derive probability densities in fully developed turbulence.

In this paper we will describe and further develop a new approach [**?**] based on extremizing the Tsallis entropies []. We will first summarize what was done in [**?**] and then derive some new results. The agreement of our theoretical approach with experimentallly measured PDFs is very good.

2 The extremum principle

Consider a suitable observable in a fully developed turbulent flow in 3 dimensions. For example, this may be a longitudinal or transversal velocity difference, a temperature or a pressure difference. The dynamics is effectively described by some highly nonlinear set of model equations—e.g. the Navier Stokes equation for velocity differences. Since nobody is able to solve the problem exactly, let us, for the moment, just try to find some effective statistical description using methods from generalized statistical mechanics.

The idea is quite similar to going from classical mechanics to ordinary thermodyanics: Although nobody was able to 'solve' the classical N-body problem with $N \geq 3$, one still was able to develop a successful effective theory of a gas of 10^{23} particles by extremizing the Boltzmann Gibbs entropy. For turbulent (non-equilibrium) systems, ordinary Botzmann Gibbs statistics is not sufficient to describe the stationary state. But still we could try to develop a theory using more general information measures.

Here we will work with the Tsallis entropies

$$S_q = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right), \tag{1}$$

The p_i are probabilities associated with the microstates of the physical system, and q is the nonextensitivity parameter. The ordinary Boltzmann-Gibbs entropy is obtained in the limit $q \to 1$.

Extremizing S_q under norm and energy constraints, one arrives at the generalized canonical distributions

$$p_i = \frac{1}{Z_q}(1 + (q-1)\beta\epsilon_i)^{\frac{1}{1-q}}, \tag{2}$$

where

$$Z_q = \sum_i (1 - (1-q)\beta\epsilon_i)^{\frac{1}{1-q}} \tag{3}$$

is the partition function, $\beta = 1/(kT)$ is a suitable inverse temperature variable, and the ϵ_i is the energy of the microstate i. One could also work with the escort distribution [**?**], defined by $P_i = p_i^q / sumpi^q$. If β is allowed to depend on q, this new distribution can also be written in the above form, with a $q/(q-1) =: 1/(q'-1)$.

All we have to decide now is what we should take for the effective energy levels ϵ_i. This depends on the problem considered. For example, turbulence in 1, 2, or 3 dimensions ought to yield different effective energy levels. Moreover, different observables will also lead to different energies. In a turbulent 3-dimensional flow for example, temperature differences should be described by different effective energies than velocity differences. This is already clear from the fact that the experimentally observed stationary probability distributions (slightly) differ.

At this stage one has to turn to some sort of model. In the following we will consider a simple model for longitudinal velocity differences that seems to be in good agreement with true turbulence experiments.

3 Perturbative approach to chaotically driven systems

Let us consider a very simple dynamical model of longitudinal velocity differences $\Delta u = u(x+r) - u(x)$ in a turbulent flow. It is based on a generalization of the Langevin approach to deterministic chaotic driving forces. In the following, we rename Δu to u. Clearly local velocity differences u relax with a certain damping constant γ and at the same time are driven by deterministic chaotic forces $F_{chaot}(t)$ in the liquid, which are very complicated. Hence a very simple local model is

$$\dot{u} = -\gamma u + F_{chaot}(t). \tag{4}$$

The force $F_{chaot}(t)$ is not Gaussian white noise but something deterministic chaotic. It changes on a typical time scale τ, which is usually smaller than the relaxation time γ^{-1}, so $\gamma\tau$ is a small parameter. Discretizing in time we may use as an even simpler model

$$F_{chaot}(t) = (\gamma\tau)^{1/2} \sum_{n=0}^{\infty} x_n \delta(t - n\tau), \tag{5}$$

where the x_n are the iterates of an appropriate stroboscopic chaotic map T. Integrating eq. (4) one obtains the following map of skew product form

$$\begin{aligned} x_{n+1} &= T(x_n) \\ u_{n+1} &= \lambda u_n + \sqrt{\gamma\tau}\, x_{n+1}, \end{aligned} \tag{6}$$

where $u_n := u(n\tau + 0)$ and $\lambda := e^{-\gamma\tau}$. A lot of work has been done on these types of maps (see, e.g. [18] for a review). In particular, it has been rigorously proved that if T is chosen to be the Ulam map $T(x) = 1 - 2x^2$, then for small but finite $\gamma\tau$ the invariant density $p(u)$ of the variable u has the slightly asymmetric form

$$p(u) = \sqrt{\frac{2}{\pi}} e^{-2u^2 + \sqrt{\gamma\tau}(2u - \frac{8}{3}u^3)} + O(\gamma\tau) \tag{7}$$

The proof is based on the analysis of higher-order correlation functions and subsequent Fourier transformation [19], or, alternatively, a perturbative expansion of the Perron-Frobenius equation [20]. If the variable u is rescaled such that the probability distribution has variance 1 (achieved by introducing the new variable $u' = 2u$ and renaming u' to u), the above result is equivalent to

$$p(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2 + c\sqrt{\gamma\tau}(u - \frac{1}{3}u^3)} + O(\gamma\tau) \tag{8}$$

with $c = 1$. An important observation is that this formula appears to be universal. In [21] numerical and theoretical evidence has been presented that not only for the special example $T(x) = 1 - 2x^2$ but for large classes of chaotic maps T with vanishing mean one always obtains the same result. Up to a non-universal constant c of $O(1)$ multiplying the term $\sqrt{\gamma\tau}(u - \frac{1}{3}u^3)$, the slightly asymmetric corrections to a Gaussian of variance 1 are always observed to be of the above form (see [21] for details). It is thus reasonable to conjecture that also the chaotic forces acting locally in a fully developed turbulent flow lie in this universality class, and we can use the above approach to contruct effective energy levels ϵ_i.

4 1st-order generalized thermostatistics for turbulent flows

Eq.(8) formally corresponds to a Boltzmann factor

$$p_i = \frac{1}{Z} e^{-\beta\epsilon_i} \tag{9}$$

with $\beta = 1/(kT) = 1$ by construction, $Z = \sqrt{2\pi}$, and energy ϵ_i formally given by

$$\epsilon_i = \frac{1}{2}u^2 - c\sqrt{\gamma\tau}(u - \frac{1}{3}u^3). \tag{10}$$

To proceed from ordinary Boltzmann-Gibbs statistical mechanics to generalized thermostatistics we have to replace the Boltzmann factor by the generalized canonical distributions of order q keeping the same energies ϵ_i. Moreover, for the application to fully developed turbulence we have to clarify the meaning of the small dimensionless parameter $\gamma\tau$. We will identify it with the inverse Taylor scale Reynolds number R_λ. To see this, one should remember that γ essentially stands for the kinematic viscosity ν. We have $\gamma = \nu/L^2$ where L is a suitable unit of length. Moreover, the time scale τ can be written as $\tau = L/V$, where V is a typical velocity. It is most reasonable to identify $V = \sqrt{\langle u^2 \rangle}$ and $L = \lambda$ (Taylor microscale). We then end up with $\gamma\tau \sim \frac{\nu}{VL} \sim R_\lambda^{-1}$.

The result of our consideration is that as a prediction for turbulent velocity distributions, using generalized thermostatistics and the universal skewness corrections, we obtain the formula

$$p(u) = \frac{1}{Z_q}\left(1 + \beta(q-1)\left(\frac{1}{2}u^2 - \frac{c}{\sqrt{R_\lambda}}(u - \frac{1}{3}u^3)\right)\right)^{-\frac{1}{q-1}}. \qquad (11)$$

where the parameter β is determined by the condition that the distribution should have variance 1. For large R_λ this is achieved for $\beta = 2/(5-3q)$. The constant c will be set to 1 in the following.

5 Physical interpretation of the parameter q

Still we have to clarify the physical meaning of the parameter q. It clearly must depend on the dimensionless distance variable r/η (η denotes the Kolmogorov scale). Let us again consider the variable $k = \frac{1}{q-1}$. For $k \to \infty$ we have Gaussian behavior, for $k = O(1)$ we have very strong deviations from Gaussian behaviour, and we are near to the Kolmogorov scale (see Fig. 1). The cascade-like structure of turbulent dynamics suggests to interpret k as the level of a cascade where the distance r doubles in each step (see also [22]). $k = O(1)$ corresponds to the Kolmogorov scale, $k = \infty$ to a cascade of infinite size. Inbetween we have $r = 2^{k-O(1)}\eta$ or

$$k = \frac{1}{q-1} = O(1) + \log_2 \frac{r}{\eta}. \qquad (12)$$

Eq. (12), together with eq. (11), yields a concrete formula for probability densities of velocity differences u measured in a flow with Reynolds number R_λ at distance r/η, which can be compared with experiments. Setting $O(1) = 1$ the formula is in excellent agreement with experimental data for turbulent flows. As an example we have chosen the published data of Castaing, Gange and Hopfinger [16] obtained for a jet experiment at $R_\lambda = 852$. Fig. 2 shows their experimentally measured velocity distributions for various r/η and our theoretical prediction for various k given by the nearest integer in eq. (12).

For better visibility, the distributions are shifted by one unit along the y-axis. Note that no free parameters are fitted for the comparison with the experiment. The Reynolds number R_λ and the distance r/η are just put into the formula. Also the data of other experiments [15, 17] are in good agreement with our formula.

6 Scaling exponents

We may also evaluate the moment scaling exponents ζ_m [23]–[29] defined by $\langle |u|^m \rangle \sim r^{\zeta_m}$ in the inertial range. In the limit $R_\lambda \to \infty$ the moments are easily evaluated [14] to give

$$\langle |u|^m \rangle = \int_{-\infty}^{\infty} p(u)|u|^m du = \left(\frac{2k}{\beta}\right)^{\frac{m}{2}} \frac{B\left(\frac{m+1}{2}, k - \frac{m+1}{2}\right)}{B\left(\frac{1}{2}, k - \frac{1}{2}\right)} \tag{13}$$

If $k = 1/(q-1)$ is an integer this can also be written as

$$\langle u^m \rangle = \left(\frac{2k}{\beta}\right)^{\frac{m}{2}} \frac{(m-1)!!}{(2k-3)(2k-5)\cdots(2k-m-1)} \tag{14}$$

for m even, respectively

$$\langle |u|^m \rangle = \left(\frac{2k}{\beta}\right)^{\frac{m}{2}} \frac{2^{k-1}\left(\frac{m-1}{2}\right)!\left(k - \frac{m+1}{2} - 1\right)!}{\pi(2k-3)!!} \tag{15}$$

for m odd. Only moments with $m < 2k - 1$ exist for a given k.

Note that in the relation $\langle |u|^m \rangle \sim r^{\zeta_m}$ the symbol $\langle \cdots \rangle$ corresponds to different probability densities, since by increase of r the index q (or k) changes. Now consider the doubling transformation of distance

$$r \to 2r. \tag{16}$$

This transformation implies a change of the index k of the density function, which, according to eq. (12), is given by

$$k \to k + 1. \tag{17}$$

Using eq. (14) or (15) we can evaluate the ratios of moments at level $k+1$ and k as

$$\frac{\langle |u|^m \rangle_{k+1}}{\langle |u|^m \rangle_k} = \left(\frac{\beta_k}{\beta_{k+1}} \left(1 + \frac{1}{k}\right)\right)^{\frac{m}{2}} \left(1 - \frac{m}{2k-1}\right). \tag{18}$$

Here β_k denotes the variance parameter β at level k. Eq. (18) is valid for both even and odd m. Since we also have from (16)

$$\frac{\langle |u|^m \rangle_{k+1}}{\langle |u|^m \rangle_k} = 2^{\zeta_m} \tag{19}$$

we obtain for the scaling exponents the formula

$$\zeta_m = \frac{m}{2} \left(\log_2 \frac{\beta_k}{\beta_{k+1}} + \log_2 \left(1 + \frac{1}{k}\right)\right) + \log_2 \left(1 - \frac{m}{2k-1}\right). \tag{20}$$

The Navier-Stokes equation in 3 dimensions is generally believed to imply $\zeta_3 = 1$. Putting $\zeta_3 = 1$ into eq. (20) we obtain a relation between β_k/β_{k+1} and k.

$$\frac{\beta_k}{\beta_{k+1}} = \frac{2^{\frac{2}{3}}}{\left(1 + \frac{1}{k}\right) \left(1 - \frac{3}{2k-1}\right)^{\frac{2}{3}}} \tag{21}$$

This yields as a final result for the ζ_m

$$\zeta_m = \frac{m}{3} \left(1 - \log_2 \left(1 - \frac{3}{2k-1}\right)\right) + \log_2 \left(1 - \frac{m}{2k-1}\right). \tag{22}$$

Note that one obtains linear behavior in m plus a nonlinear correction term, similar as in other models [23]–[29]. In a typical turbulence experiment, the inertial range is usually observed to be a rather small region corresponding to r/η of the order 30...130 [15]. Since, for a given k, our formula describes effective scaling exponents in the region $[2^{k-1}\eta, 2^k\eta]$ this means that typically just $k = 6$ and $k = 7$ are of relevance, and we may form a suitable average. Table 1 shows that formula (22) yields very good coincidence with experimentally measured ζ_m within the experimental uncertainties, which are of the order 10% for $m = 8$. It does not make sense to go to very large m, since in experimental measurements the fluctuations become huge, and in our theoretical approach very high moments ($m \geq 2k-1$) do not exist.

If one plots the logarithm of the moments as a function of $\log r/\eta$ one gets pictures very similar to experimentally measured structure functions as published, e.g., in [15]. Instead of exact straight lines one obtaines slightly curved lines, precisely as observed in experiments.

7 Conclusion

Summarizing, all our theoretical formulas are in very good agreement with experimental measurements for hydrodynamical turbulent flows. This indicates that fully developed turbulent flows indeed seem to extremize the Tsallis entropies, at least in a very good approximation. The generalized thermostatistics with $q \neq 1$ yields a new useful theoretical approach, producing explicit formulas which can be checked by experiments. The parameter q is not constant but depends on distance. Our approach is also applicable to describe the statistics of other turbulent-like phenomena (such as share price or currency fluctuations) which clearly do not obey the Navier-Stokes equations but exhibit similar scaling phenomena [30]–[32].

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References

[1] C. Tsallis, J. Stat. Phys. **52**, 479 (1988)

[2] C. Tsallis, R.S. Mendes and A.R. Plastino, Physica **261A**, 534 (1998)

[3] C. Tsallis, Braz. J. Phys. **29**, 1 (1999)

[4] A. Rényi, *Probability Theory*, North-Holland, Amsterdam (1970)

[5] C. Beck and F. Schlögl, *Thermodynamics of Chaotic Systems*, Camdridge University Press, Cambridge (1993)

[6] P. Jund, S.G. Kim and C. Tsallis, Phys. Rev. **52B**, 50 (1995)

[7] Y.-H. Taguchi and H. Takayasu, Europhys. Lett. **30**, 499 (1995)

[8] M.L. Lyra and C. Tsallis, Phys. Rev. Lett. **80**, 53 (1998)

[9] X.-P. Huang and C.F. Driscoll, Phys. Rev. Lett. **72**, 2187 (1994)

[10] B.M. Boghosian, Phys. Rev. **53E**, 4754 (1996)

[11] A. Lavagno, G. Kaniadakis, M. Rego-Monteiro, P. Quarati and C. Tsallis, Astrophys. Lett. and Comm. **35**, 449 (1998)

[12] I. Bediaga, E.M.F. Curado and J. Miranda, hep-ph/9905 255 (1999)

[13] F.M. Ramos, R.R. Rosa and C. Rodrigues Neto, cond-mat/9907 348 (1999)

[14] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York (1965)

[15] F. Anselmet, Y. Gagne, E.J. Hopfinger and R.A. Antonia, J. Fluid Mech. **140**, 63 (1984)

[16] B. Castaing, Y. Gagne and E.J. Hopfinger, Physica **46D**, 177 (1990)

[17] B. Chabaud, A. Naert, J. Peinke, F. Chilla, B. Castaing and B. Hebral, Phys. Rev. Lett. **73**, 3227 (1994)

[18] C. Beck, Physica **233A**, 419 (1996)

[19] C. Beck, Nonlinearity **4**, 1131 (1991)

[20] C. Beck, J. Stat. Phys. **79**, 875 (1995)

[21] A. Hilgers and C. Beck, Phys. Rev. **60E**, 5385 (1999)

[22] C. Beck, Phys. Rev. **49E**, 3641 (1994)

[23] A.N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962)

[24] V.S. Lvov and I. Procaccia, Physica **257A**, 165 (1998)

[25] Z.-S. She and E. Leveque, Phys. Rev. Lett. **72**, 336 (1994)

[26] Ch. Meneveau and K.R. Sreenivasan, J. Fluid. Mech. **224**, 429 (1991)

[27] R. Benzi, S. Ciliberto, C. Baudet and G.R. Chavarria, Physica **80D**, 385 (1995)

[28] U. Frisch, *Turbulence: the Legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge (1995)

[29] T. Bohr, M.H. Jensen, G. Paladin and A. Vulpiani, *Dynamical Systems Approach to Turbulence*, Cambridge University Press, Cambridge (1998)

[30] R.N. Mantegna and H.E. Stanley, Nature **376**, 46 (1995)

[31] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner and Y. Dodge, Nature **381**, 767 (1996)

[32] A. Hilgers and C. Beck, Int. J. Bif. Chaos **7**, 1855 (1997)

Table 1:
Scaling exponents ζ_m as obtained from Eq. (22) and comparison with experimentally measured values.

m	Theory ($k=6$)	Theory ($k=7$)	Average	Exp.[15]	Exp.[27]
2	0.68	0.68	0.68	0.71	0.70
3	1	1	1	1.00	1.00
4	1.29	1.31	1.30	1.33	1.28
5	1.56	1.60	1.58	1.65	1.54
6	1.78	1.86	1.82	1.80	1.78
7	1.95	2.10	2.02	2.12	2.00
8	2.02	2.30	2.16	2.22	2.23

Fig. 1 Probability densities $p(u)$ of variance 1 as given by eq. (**??**) for $k = \frac{1}{q-1} = 3, 4, \ldots, 20$ (from top to bottom). For better visibility each distribution is shifted by 0.1 units along the y-axis.

Fig. 2 Experimentally measured probability densities $p(u)$ of variance 1 for a jet experiment at $R_\lambda = 852$ (data from [16]) and the theoretical prediction (11). From top to bottom the distance is $r/\eta = 3.3(\diamond)$, $r/\eta = 23.6(+)$, $r/\eta = 100(\Box)$. Eq.(11) is plotted for $k = \frac{1}{q-1} = 3, 5, 7$.