Application of generalized thermostatistics to fully developed turbulence

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Abstract

We apply the formalism of non-extensive statistical mechanics to fully developed turbulent flows. We obtain analytical formulas for probability density functions of velocity differences depending on distance r/η and Reynolds number R_λ, which are in very good agreement with turbulence experiments. The skewness is taken into account by analyzing universal contributions due to higher-order correlations of the chaotic dynamics. We also calculate the moment scaling exponents ζ_m, which also agree well with experimental data.

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1 Introduction

Recently, the introduction of a generalized version of statistical mechanics has gained considerable interest [1]–[3]. In this approach, the usual Boltzmann-Gibbs entropy $S = -\sum_i p_i \ln p_i$ is replaced by the more general Tsallis entropies

$$S_q = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right), \tag{1}$$

which are closely related to the Rényi information measures [4]. The p_i are probabilities associated with the microstates of a physical system, and $q \in (-\infty, \infty)$ is a real parameter. The ordinary Boltzmann-Gibbs entropy is obtained in the limit $q \to 1$.

Extremizing S_q under the norm constraint

$$\sum_i p_i = 1 \tag{2}$$

and the energy constraint

$$\sum_i P_i \epsilon_i = U_q, \tag{3}$$

where the ϵ_i are the energy levels of the microstates, and

$$P_i = \frac{p_i^q}{\sum_i p_i^q} \tag{4}$$

denote the so-called escort distributions [5], one formally obtains a generalized version of the canonical ensemble. The probabilities p_i come out of the extremization procedure as

$$p_i = \frac{1}{Z_q}(1 - (1-q)\beta\epsilon_i)^{\frac{1}{1-q}}, \tag{5}$$

where

$$Z_q = \sum_i (1 - (1-q)\beta\epsilon_i)^{\frac{1}{1-q}} \tag{6}$$

is the partition function and $\beta = 1/(kT)$ is a suitable inverse temperature variable (denoted as β' in [3]). In the limit $q \to 1$, ordinary statistical mechanics is recovered.

The remarkable fact is that several physical systems have recently been found that are correctly described by the generalized thermostatistics with $q \neq 1$. In the old Boltzmann-Gibbs formalism some of these systems could actually be regarded as non-equilibrium systems, though with a quasi-stationary state. Applications have been reported for Hamiltonian systems with long-range interactions [6], granular systems [7], systems with strange non-chaotic attractors [8] and quite generally multifractal systems. Concrete experimental evidence has been reported for $D = 2$ turbulence in pure electron plasmas [9, 10], peculiar velocities in Sc galaxies [11], scattering data in particle physics [12] and many other systems.

In this paper we will apply the new formalism to fully developed hydrodynamical turbulence in 3 dimensions (see also [13] for previous attempts using several fitting parameters). We want to see what the predictions of generalized thermostatistics are. In particular, we will build skewness into the theory, which is necessary for an understanding of the slightly asymmetric probability distributions measured in turbulence experiments. We will obtain analytical formulas for probability densities and scaling exponents that are in very good agreement with experimental measurements.

2 0th-order generalized thermostatistics for turbulent flows

To apply the generalized thermostatistics, we first have to identify the (formal) energy levels ϵ_i in our case. The simplest approach is to relate them to the kinetic energy $\frac{1}{2}u^2$ associated with the radial velocity difference u between two points in the liquid separated by a distance r. Extremizing the Tsallis entropy S_q one obtains for the probability density $p(u)$ the formula

$$p(u) = \frac{1}{Z_q}\left(1 + \frac{1}{2}(q-1)\beta u^2\right)^{-\frac{1}{q-1}}, \tag{7}$$

where

$$Z_q = \int_{-\infty}^{+\infty}\left(1 + \frac{1}{2}(q-1)\beta u^2\right)^{-\frac{1}{q-1}} du \tag{8}$$

is the partition function. The integral exists for $1 \leq q < 3$ and can be evaluated explicitly [14] to give

$$Z_q = \sqrt{\frac{2}{(q-1)\beta}} \cdot B\left(\frac{1}{2}, \frac{1}{q-1} - \frac{1}{2}\right). \tag{9}$$

Here

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{10}$$

denotes the beta-function. If $k := \frac{1}{q-1} \geq 2$ is an integer, we may further evaluate Z_q as

$$Z_q = \sqrt{\frac{2k}{\beta}} \frac{\pi(2k-3)!!}{(k-1)!2^{k-1}}. \tag{11}$$

Changing k from large values to smaller values, the above probability densities exhibit a transition scenario from almost Gaussian distributions to stretched distributions (Fig. 1). The scenario is quite similar to what is observed in turbulence experiments [15]–[17]. However, an important property of turbulent velocity distributions is still missing so far, namely the skewness (the slight asymmetry) of the experimentally measured distributions.

3 Skewness

To incorporate skewness, let us start with a very simple dynamical model of local velocity differences u in a turbulent flow. Clearly local velocity differences relax with a certain damping constant γ and at the same time are driven by deterministic chaotic forces $F_{chaot}(t)$ in the liquid, which are very complicated. Hence a very simple local model is

$$\dot{u} = -\gamma u + F_{chaot}(t). \tag{12}$$

The force $F_{chaot}(t)$ is not Gaussian white noise but something deterministic chaotic. It changes on a typical time scale τ, which is usually smaller than the relaxation time γ^{-1}, so $\gamma\tau$ is a small parameter. Discretizing in time we may use as an even simpler model

$$F_{chaot}(t) = (\gamma\tau)^{1/2} \sum_{n=0}^{\infty} x_n \delta(t - n\tau), \tag{13}$$

where the x_n are the iterates of an appropriate stroboscopic chaotic map T. Integrating eq. (12) one obtains the following map of skew product form

$$\begin{aligned} x_{n+1} &= T(x_n) \\ u_{n+1} &= \lambda u_n + \sqrt{\gamma\tau}\, x_{n+1}, \end{aligned} \tag{14}$$

where $u_n := u(n\tau + 0)$ and $\lambda := e^{-\gamma\tau}$. A lot of work has been done on these types of maps (see, e.g. [18] for a review). In particular, it has been rigorously proved that if T is chosen to be the Ulam map $T(x) = 1 - 2x^2$, then for small but finite $\gamma\tau$ the invariant density $p(u)$ of the variable u has the slightly asymmetric form

$$p(u) = \sqrt{\frac{2}{\pi}} e^{-2u^2 + \sqrt{\gamma\tau}(2u - \frac{8}{3}u^3)} + O(\gamma\tau) \tag{15}$$

The proof is based on the analysis of higher-order correlation functions and subsequent Fourier transformation [19], or, alternatively, a perturbative expansion of the Perron-Frobenius equation [20]. If the variable u is rescaled such that the probability distribution has variance 1 (achieved by introducing the new variable $u' = 2u$ and renaming u' to u), the above result is equivalent to

$$p(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2 + c\sqrt{\gamma\tau}(u - \frac{1}{3}u^3)} + O(\gamma\tau) \tag{16}$$

with $c = 1$. An important observation is that this formula appears to be universal. In [21] numerical and theoretical evidence has been presented that not only for the special example $T(x) = 1 - 2x^2$ but for large classes of chaotic maps T with vanishing mean one always obtains the same result. Up to a non-universal constant c of $O(1)$ multiplying the term $\sqrt{\gamma\tau}(u - \frac{1}{3}u^3)$, the slightly asymmetric corrections to a Gaussian of variance 1 are always observed to be of the above form (see [21] for details). It is thus reasonable to conjecture that also the chaotic forces acting locally in a fully developed turbulent flow lie in this universality class, and we can use the above approach to contruct effective energy levels ϵ_i.

4 1st-order generalized thermostatistics for turbulent flows

Eq.(16) formally corresponds to a Boltzmann factor

$$p_i = \frac{1}{Z} e^{-\beta \epsilon_i} \tag{17}$$

with $\beta = 1/(kT) = 1$ by construction, $Z = \sqrt{2\pi}$, and energy ϵ_i formally given by

$$\epsilon_i = \frac{1}{2}u^2 - c\sqrt{\gamma\tau}(u - \frac{1}{3}u^3). \tag{18}$$

To proceed from ordinary Boltzmann-Gibbs statistical mechanics to generalized thermostatistics we have to replace the Boltzmann factor by the generalized canonical distributions of order q keeping the same energies ϵ_i. Moreover, for the application to fully developed turbulence we have to clarify the meaning of the small dimensionless parameter $\gamma\tau$. We will identify it with the inverse Taylor scale Reynolds number R_λ. To see this, one should remember that γ essentially stands for the kinematic viscosity ν. We have $\gamma = \nu/L^2$ where L is a suitable unit of length. Moreover, the time scale τ can be written as $\tau = L/V$, where V is a typical velocity. It is most reasonable to identify $V = \sqrt{\langle u^2 \rangle}$ and $L = \lambda$ (Taylor microscale). We then end up with $\gamma\tau \sim \frac{\nu}{VL} \sim R_\lambda^{-1}$.

The result of our consideration is that as a prediction for turbulent velocity distributions, using generalized thermostatistics and the universal skewness corrections, we obtain the formula

$$p(u) = \frac{1}{Z_q} \left(1 + \beta(q-1) \left(\frac{1}{2}u^2 - \frac{c}{\sqrt{R_\lambda}}(u - \frac{1}{3}u^3) \right) \right)^{-\frac{1}{q-1}}. \tag{19}$$

where the parameter β is determined by the condition that the distribution should have variance 1. For large R_λ this is achieved for $\beta = 2/(5-3q)$. The constant c will be set to 1 in the following.

5 Physical interpretation of the parameter q

Still we have to clarify the physical meaning of the parameter q. It clearly must depend on the dimensionless distance variable r/η (η denotes the Kol-

mogorov scale). Let us again consider the variable $k = \frac{1}{q-1}$. For $k \to \infty$ we have Gaussian behavior, for $k = O(1)$ we have very strong deviations from Gaussian behaviour, and we are near to the Kolmogorov scale (see Fig. 1). The cascade-like structure of turbulent dynamics suggests to interpret k as the level of a cascade where the distance r doubles in each step (see also [22]). $k = O(1)$ corresponds to the Kolmogorov scale, $k = \infty$ to a cascade of infinite size. Inbetween we have $r = 2^{k-O(1)}\eta$ or

$$k = \frac{1}{q-1} = O(1) + \log_2 \frac{r}{\eta}. \tag{20}$$

Eq. (20), together with eq. (19), yields a concrete formula for probability densities of velocity differences u measured in a flow with Reynolds number R_λ at distance r/η, which can be compared with experiments. Setting $O(1) = 1$ the formula is in excellent agreement with experimental data for turbulent flows. As an example we have chosen the published data of Castaing, Gange and Hopfinger [16] obtained for a jet experiment at $R_\lambda = 852$. Fig. 2 shows their experimentally measured velocity distributions for various r/η and our theoretical prediction for various k given by the nearest integer in eq. (20). For better visibility, the distributions are shifted by one unit along the y-axis. Note that no free parameters are fitted for the comparison with the experiment. The Reynolds number R_λ and the distance r/η are just put into the formula. Also the data of other experiments [15, 17] are in good agreement with our formula.

6 Scaling exponents

We may also evaluate the moment scaling exponents ζ_m [23]–[29] defined by $\langle |u|^m \rangle \sim r^{\zeta_m}$ in the inertial range. In the limit $R_\lambda \to \infty$ the moments are easily evaluated [14] to give

$$\langle |u|^m \rangle = \int_{-\infty}^{\infty} p(u)|u|^m du = \left(\frac{2k}{\beta}\right)^{\frac{m}{2}} \frac{B\left(\frac{m+1}{2}, k - \frac{m+1}{2}\right)}{B\left(\frac{1}{2}, k - \frac{1}{2}\right)} \tag{21}$$

If $k = 1/(q-1)$ is an integer this can also be written as

$$\langle u^m \rangle = \left(\frac{2k}{\beta}\right)^{\frac{m}{2}} \frac{(m-1)!!}{(2k-3)(2k-5)\cdots(2k-m-1)} \tag{22}$$

for m even, respectively

$$\langle |u|^m \rangle = \left(\frac{2k}{\beta}\right)^{\frac{m}{2}} \frac{2^{k-1}\left(\frac{m-1}{2}\right)!\left(k-\frac{m+1}{2}-1\right)!}{\pi(2k-3)!!} \tag{23}$$

for m odd. Only moments with $m < 2k-1$ exist for a given k.

Note that in the relation $\langle |u|^m \rangle \sim r^{\zeta_m}$ the symbol $\langle \cdots \rangle$ corresponds to different probability densities, since by increase of r the index q (or k) changes. Now consider the doubling transformation of distance

$$r \to 2r. \tag{24}$$

This transformation implies a change of the index k of the density function, which, according to eq. (20), is given by

$$k \to k+1. \tag{25}$$

Using eq. (22) or (23) we can evaluate the ratios of moments at level $k+1$ and k as

$$\frac{\langle |u|^m \rangle_{k+1}}{\langle |u|^m \rangle_k} = \left(\frac{\beta_k}{\beta_{k+1}}\left(1+\frac{1}{k}\right)\right)^{\frac{m}{2}} \left(1-\frac{m}{2k-1}\right). \tag{26}$$

Here β_k denotes the variance parameter β at level k. Eq. (26) is valid for both even and odd m. Since we also have from (24)

$$\frac{\langle |u|^m \rangle_{k+1}}{\langle |u|^m \rangle_k} = 2^{\zeta_m} \tag{27}$$

we obtain for the scaling exponents the formula

$$\zeta_m = \frac{m}{2}\left(\log_2 \frac{\beta_k}{\beta_{k+1}} + \log_2\left(1+\frac{1}{k}\right)\right) + \log_2\left(1-\frac{m}{2k-1}\right). \tag{28}$$

The Navier-Stokes equation in 3 dimensions is generally believed to imply $\zeta_3 = 1$. Putting $\zeta_3 = 1$ into eq. (28) we obtain a relation between β_k/β_{k+1} and k.

$$\frac{\beta_k}{\beta_{k+1}} = \frac{2^{\frac{2}{3}}}{\left(1+\frac{1}{k}\right)\left(1-\frac{3}{2k-1}\right)^{\frac{2}{3}}} \tag{29}$$

This yields as a final result for the ζ_m

$$\zeta_m = \frac{m}{3}\left(1 - \log_2\left(1 - \frac{3}{2k-1}\right)\right) + \log_2\left(1 - \frac{m}{2k-1}\right). \qquad (30)$$

Note that one obtains linear behavior in m plus a nonlinear correction term, similar as in other models [23]–[29]. In a typical turbulence experiment, the inertial range is usually observed to be a rather small region corresponding to r/η of the order 30...130 [15]. Since, for a given k, our formula describes effective scaling exponents in the region $[2^{k-1}\eta, 2^k\eta]$ this means that typically just $k = 6$ and $k = 7$ are of relevance, and we may form a suitable average. Table 1 shows that formula (30) yields very good coincidence with experimentally measured ζ_m within the experimental uncertainties, which are of the order 10% for $m = 8$. It does not make sense to go to very large m, since in experimental measurements the fluctuations become huge, and in our theoretical approach very high moments ($m \geq 2k - 1$) do not exist.

If one plots the logarithm of the moments as a function of $\log r/\eta$ one gets pictures very similar to experimentally measured structure functions as published, e.g., in [15]. Instead of exact straight lines one obtaines slightly curved lines, precisely as observed in experiments.

7 Conclusion

Summarizing, all our theoretical formulas are in very good agreement with experimental measurements for hydrodynamical turbulent flows. This indicates that fully developed turbulent flows indeed seem to extremize the Tsallis entropies, at least in a very good approximation. The generalized thermostatistics with $q \neq 1$ yields a new useful theoretical approach, producing explicit formulas which can be checked by experiments. The parameter q is not constant but depends on distance. Our approach is also applicable to describe the statistics of other turbulent-like phenomena (such as share price or currency fluctuations) which clearly do not obey the Navier-Stokes equations but exhibit similar scaling phenomena [30]–[32].

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Table 1:
Scaling exponents ζ_m as obtained from Eq. (30) and comparison with experimentally measured values.

m	Theory ($k=6$)	Theory ($k=7$)	Average	Exp.[15]	Exp.[27]
2	0.68	0.68	0.68	0.71	0.70
3	1	1	1	1.00	1.00
4	1.29	1.31	1.30	1.33	1.28
5	1.56	1.60	1.58	1.65	1.54
6	1.78	1.86	1.82	1.80	1.78
7	1.95	2.10	2.02	2.12	2.00
8	2.02	2.30	2.16	2.22	2.23

Fig. 1 Probability densities $p(u)$ of variance 1 as given by eq. (7) for $k = \frac{1}{q-1} = 3, 4, \ldots, 20$ (from top to bottom). For better visibility each distribution is shifted by 0.1 units along the y-axis.

Fig. 2 Experimentally measured probability densities $p(u)$ of variance 1 for a jet experiment at $R_\lambda = 852$ (data from [16]) and the theoretical prediction (19). From top to bottom the distance is $r/\eta = 3.3(\diamond)$, $r/\eta = 23.6(+)$, $r/\eta = 100(\square)$. Eq.(19) is plotted for $k = \frac{1}{q-1} = 3, 5, 7$.