Fluctuation Relations for Anomalous Dynamics Generated by Time Fractional Fokker-Planck Equations

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Motivation: Fluctuation relations

Consider a (classical) particle system evolving from some initial state into a nonequilibrium steady state.
Measure the probability distribution $\rho(\xi_t)$ of entropy production ξ_t during time t:

$$\ln \frac{\rho(\xi_t)}{\rho(-\xi_t)} = \xi_t$$

Transient Fluctuation Relation (TFR)

Evans, Cohen, Morriss (1993); Gallavotti, Cohen (1995)

why important? of *very general validity* and

1. generalizes the Second Law to small systems in nonequ.
2. connection with fluctuation dissipation relations (FDRs)
3. can be checked in experiments (Wang et al., 2002)

Anomalous TFR for Gaussian stochastic processes

known result:

consider **overdamped generalized Langevin equation**

$$\dot{x} = F + \zeta(t)$$

with force F and Gaussian power-law correlated noise

$$< \zeta(t)\zeta(t') >_{\tau=t-t'} \sim (\Delta/\tau)^{\beta} \text{ for } \tau > \Delta\ ,\ \beta > 0$$

that is external (i.e., no FDR):

- dynamics can generate **anomalous diffusion**, $\sigma_x^2 \sim t^{2-\beta}$ with $2-\beta \neq 1\ (t \to \infty)$
- yields an **anomalous work fluctuation relation**,

$$\ln \frac{\rho(W_t)}{\rho(-W_t)} = \mathbf{f}_{\beta}(\mathbf{t}) W_t$$

A.V.Chechkin et al., J.Stat.Mech. L11001 (2012) and L03002 (2009)

Question: what's about non-Gaussian processes?

Modeling non-Gaussian dynamics

- start again from overdamped Langevin equation $\dot{x} = F + \zeta(t)$, but here with **non-Gaussian** power law correlated noise

$$< \zeta(t)\zeta(t') >_{\tau=t-t'} \sim (K_\alpha/\tau)^{2-\alpha} \; , \; 1 < \alpha < 2$$

- 'motivates' the non-Markovian Fokker-Planck equation

type A: $\frac{\partial \varrho_A(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F - K_\alpha D_t^{1-\alpha}\frac{\partial}{\partial x}\right]\varrho_A(x,t)$

with Riemann-Liouville fractional derivative $D_t^{1-\alpha}$ (Balescu, 1997)

- two *formally similar* types derived from CTRW theory, for $0 < \alpha < 1$:

type B: $\frac{\partial \varrho_B(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F - K_\alpha D_t^{1-\alpha}\frac{\partial}{\partial x}\right]\varrho_B(x,t)$

type C: $\frac{\partial \varrho_C(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[F D_t^{1-\alpha} - K_\alpha D_t^{1-\alpha}\frac{\partial}{\partial x}\right]\varrho_C(x,t)$

They model a *different* class of stochastic process!

Properties of non-Gaussian dynamics

Riemann-Liouville fractional derivative defined by

$$\frac{\partial^\gamma \varrho}{\partial t^\gamma} := \begin{cases} \frac{\partial^m \varrho}{\partial t^m} & , \quad \gamma = m \\ \frac{\partial^m}{\partial t^m}\left[\frac{1}{\Gamma(m-\gamma)} \int_0^t dt' \frac{\varrho(t')}{(t-t')^{\gamma+1-m}}\right] & , \quad m-1 < \gamma < m \end{cases}$$

with $m \in \mathbb{N}$; power law inherited from correlation decay.

two important properties:

- **FDR:** exists for type C but not for A and B
- **mean square displacement:**

- type A: superdiffusive, $\sigma_x^2 \sim t^\alpha$, $1 < \alpha < 2$

- type B: subdiffusive, $\sigma_x^2 \sim t^\alpha$, $0 < \alpha < 1$

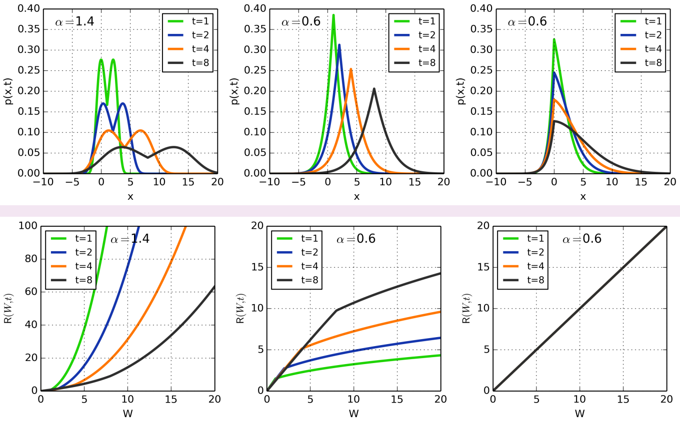
- type C: sub- or superdiffusive, $\sigma_x^2 \sim t^{2\alpha}$, $0 < \alpha < 1$

- **position pdfs:** can be calculated approximately analytically for A, B, only numerically for C

Probability distributions and fluctuation relations

type A | type B | type C

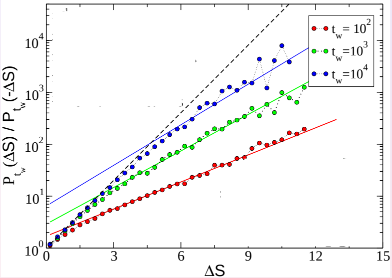
- **PDFs:**
- **TFRs:**



$$R(W_t) = \log \frac{\rho(W_t)}{\rho(-W_t)} \sim \begin{cases} c_\alpha W_t \ , & W_t \to 0 \\ t^{(2\alpha-2)/(\alpha-2)} W_t^{\alpha/(2-\alpha)} \ , & W_t \to \infty \end{cases}$$

Relations to experiments: glassy dynamics

example 1: computer simulations for a binary Lennard-Jones mixture below the glass transition

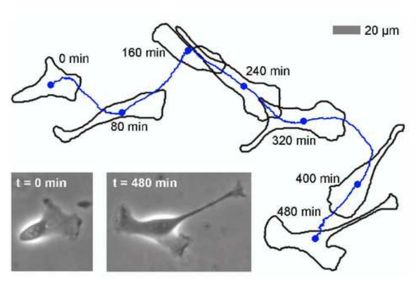


Crisanti, Ritort, PRL (2013)

- again: $R(W_t) = \ln \frac{\rho(W_t)}{\rho(-W_t)} = \mathbf{f}_\beta(\mathbf{t}) W_t$; cp. with TFR type B
- similar results for other glassy systems (Sellitto, PRE, 2009)

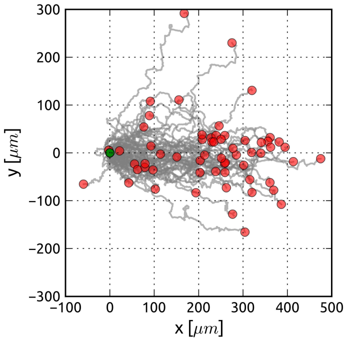
Cell migration without and with chemotaxis

example 2: single MDCKF cell crawling on a substrate; trajectory recorded with a video camera



Dieterich et al., PNAS, 2008

new experiments on murine neutrophils under chemotaxis:

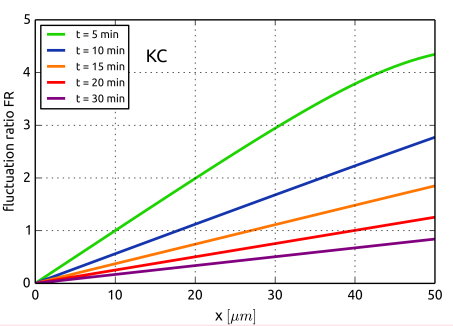


Dieterich et al. (2013)

Anomalous fluctuation relation for cell migration

preliminary **experimental results:**

- $< x(t) > \sim t$ and $\sigma_x^2 \sim t^{2-\beta}$ with $0 < \beta < 1$: $\nexists$ FDR
- fluctuation ratio $R(W_t)$ is **time dependent**:



$$R(W_t) = \ln \frac{\rho(W_t)}{\rho(-W_t)} = \frac{W_t}{\mathbf{t}^{1-\beta}}$$

Summary

TFR tested for **non-Gaussian dynamics** modeled by three cases of **time fractional Fokker Planck equations**:

- breaking FDR implies (again) anomalous TFRs
- for non-Gaussian dynamics the TFR displays a nonlinear dependence on the (work) variable, in contrast to Gaussian stochastic processes
- anomalous TFRs appear to be important for glassy ageing dynamics

References

P.Dieterich, RK, A.V. Chechkin, NJP **17**, 075004 (2015)