Dependence of chaotic diffusion on size and position of holes

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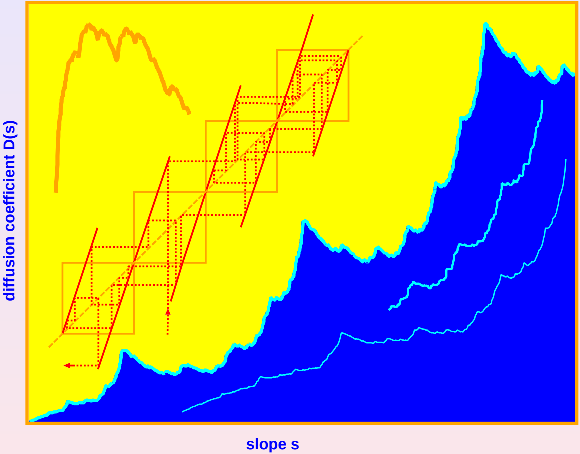
WARNING:

This talk is given by a theoretical physicist.

Outline

1. **hole dependence of diffusion** in a simple chaotic map
2. **parameter dependence of diffusion** in other maps and billiards

Simple maps with fractal diffusion coefficients



RK, Dorfman (1995)

Diffusion coefficient $D(s)$ is a fractal function of the slope s, due to topological instability under parameter variation.

Where to place a hole to achieve a max. escape rate?

Bunimovich, Yurchenko (2008/2011): **Bernoulli shift**

$$B(x) = 2x \bmod 1 \; , \; x_{n+1} = B(x_n)$$

with escape through a hole.

Theorem

Consider holes at *different positions* but with *equal size*.
Find in each hole the *periodic point with minimal period*.
Then the escape will be **faster** through the hole where the minimal period is **bigger**.

Corollary

The escape rate may be larger through smaller holes (cf. $D(s)$).

related work by Keller, Liverani (2009)

Question: Connection wrt irregularities diffusion $\leftrightarrow$ escape?

Escape rate and diffusion coefficient

Solve the **one-dimensional diffusion equation**

$$\frac{\partial \varrho}{\partial t} = D \frac{\partial^2 \varrho}{\partial x^2}$$

for particle density $\varrho = \varrho(x,t)$ and diffusion coefficient D with absorbing boundary conditions $\varrho(0,t) = \varrho(L,t) = 0$:

$$\varrho(x,t) \simeq A \exp(-\gamma t) \sin\left(\frac{\pi}{L}x\right) \quad (t, L \to \infty)$$

exponential decay with

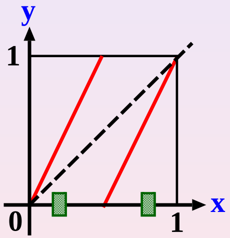
$$D = \left(\frac{L}{\pi}\right)^2 \gamma$$

escape rate γ yields **diffusion coefficient** D

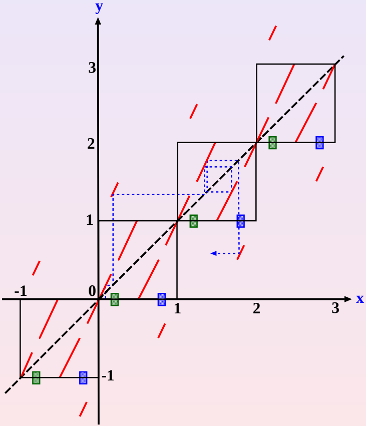
For a diffusive dynamical system the same relation can be established by solving the Frobenius-Perron equation, see *escape rate formalism* by Gaspard, Nicolis, Dorfman (1990ff)

A deterministically diffusive map

dig symmetric holes into the Bernoulli shift:



lift the unit cell by degree one and couple the cells suitably:



Question: How does the **diffusion coefficient** of this model depend on **size and position of a hole**?

Computing hole-dependent diffusion coefficients

rewrite the diffusion coefficient

$$D := \lim_{n\to\infty} \frac{< (x_n - x)^2 >}{2n}$$

with average $< \ldots >:= \int_0^1 dx\ \rho(x) \ldots\ ,\ x = x_0$ over invariant density $\rho(x)$ for $B(x)$ as

$$D_n = \frac{1}{2}\left\langle v_0^2 \right\rangle + \sum_{k=1}^n \left\langle v_0 v_k \right\rangle \to D \quad (n \to \infty)$$

Taylor-Green-Kubo formula

with integer velocities $v_k(x) = \lfloor x_{k+1} \rfloor - \lfloor x_k \rfloor$ at discrete time k

jumps between cells are captured by *fractal functions*

$$T(x) := \int_0^x d\tilde{x} \sum_{k=0}^{\infty} v_k(\tilde{x})\ ,$$

as solutions of (*de Rham-type*) functional recursion relations

Computing the (w)hole diffusion coefficient

For the Bernoulli shift $B(x)$ the invariant density is $\rho(x) = 1$.

Define the coupling by creating a map $\tilde{B}(x) : [0,1] \to [-1,2]$:

- jump through *left* hole to the *right*: if $x \in [a_1, a_2]$, $0 < a_1 < a_2 \leq 0.5$ then $\tilde{B}(x) = B(x) + 1$ yielding $v_k(x) = 1$
- jump through *right* hole to the *left*: if $x \in [1 - a_1, 1 - a_2]$ then $\tilde{B}(x) = B(x) - 1$ yielding $v_k(x) = -1$
- otherwise no jump, $\tilde{B}(x) = B(x)$ yielding $v_k(x) = 0$

This map is lifted by degree one, $\tilde{B}(x+1) = \tilde{B}(x) + 1 \, , \, x \in \mathbb{R}$.

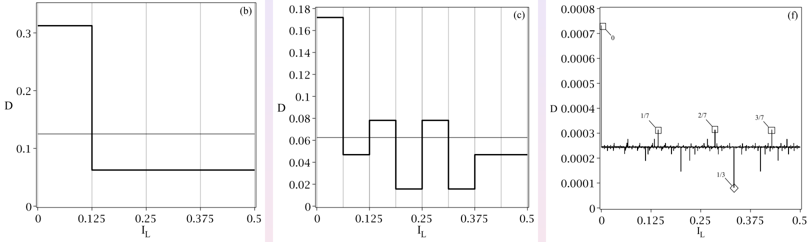
For this spatially extended model we obtain the exact result

$$D = 2T(a_2) - 2T(a_1) - h \, ; \, h = a_2 - a_1$$

Knight et al. (2011)

Diffusion coefficient vs. hole position

Diffusion coefficient D as a function of the position of the left hole I_L of size $h = a_2 - a_1 = 1/2^s$, $s = 3, 4, 12$:



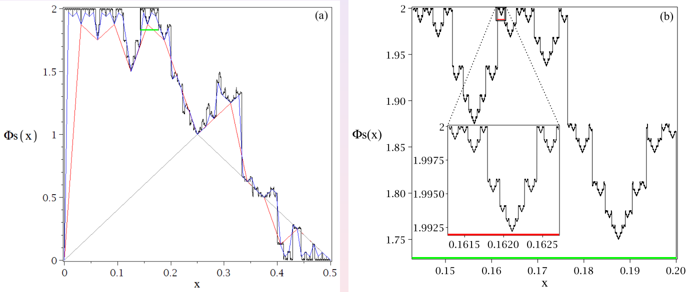
- (b), (c): for $I_L = [0.125, 0.25]$ it is $D = 1/16$, but for smaller hole $I_L = [0.125, 0.1875]$ we get larger $D = 5/64$
- (f): at $x = 0, 1/7, 2/7, 3/7$ particle keeps running through holes in one direction; at $x = 1/3$ particle jumps back and forth; these orbits *dominate diffusion* in the small hole limit

A fractal structure in the diffusion coefficient

resolve the irregular structure of the hole-dependent diffusion coefficient D by defining the cumulative function

$$\Phi_s(x) = 2^{s+1} \int_0^x (D(y) - 2^{-s})\, dy$$

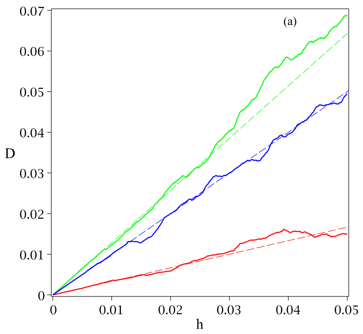
(subtract $< D_s >= 2^{-s}$ from $D(x)$ and scale with 2^{s+1})



- $\Phi_s(x)$ converges towards a fractal structure for large s
- this structure originates from the dense set of periodic orbits in $B(x)$ dominating diffusion

Diffusion for asymptotically small holes

center the hole on a standing, a non-periodic and a running orbit and let the hole size $h \to 0$:



dashed lines from analytical approximation for small h

$$D(h) \simeq \begin{cases} h\frac{1+2^{-p}}{1-2^{-p}} \,, & \text{running} \\ h\frac{1-2^{-p/2}}{1+2^{-p/2}} \,, & \text{standing} \\ h \,, & \text{non-periodic} \end{cases}$$

p : period of the orbit

- fractal parameter dependencies for $D(h)$ (RK, Dorfman, 1995)
- **violation of the random walk approximation** for small holes converging to periodic orbits

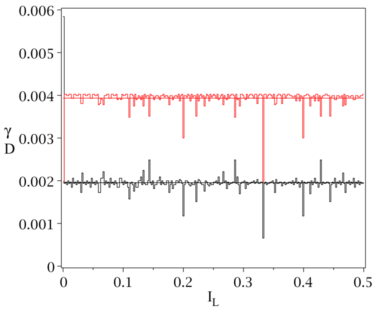
Diffusion vs. escape

γ: escape rate out of only one box
D: diffusion coefficient for the whole chain of boxes

for small h:

	running	standing
$\gamma - \langle\gamma\rangle$	$-\frac{2h}{2^p}$	$-\frac{2h}{2^{p/2}}$
$D - \langle D\rangle$	$\frac{2h}{2^p-1}$	$-\frac{2h}{2^{p/2}+1}$

p: period of orbit



$\Rightarrow \exists$ similarities and differences between γ and D

A lifted Bernoulli shift with different holes

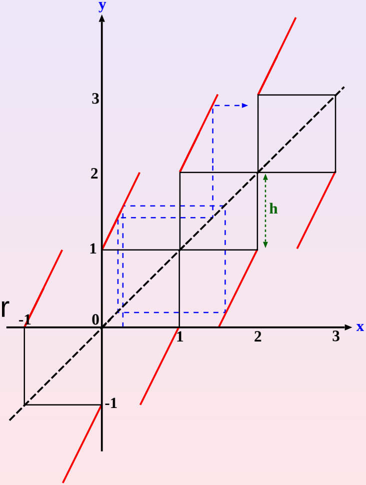
Consider the symmetrically shifted Bernoulli shift

$$M_h(x) = \begin{cases} 2x + h & 0 \le x < \frac{1}{2} \\ 2x - 1 - h & \frac{1}{2} \le x < 1 \end{cases} ,$$

$$M_h(x+1) = M_h(x) + 1$$

with shift $h \ge 0$ as a control parameter (different h cp. before, sorry!)

Gaspard, RK (1998)

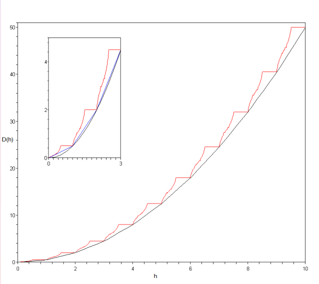


Another fractal diffusion coefficient

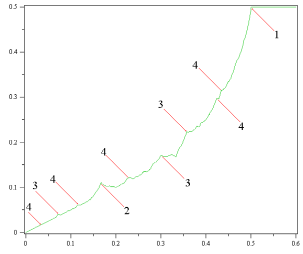
Applying the Taylor-Green-Kubo method yields

$$D(h) = \frac{\lceil h \rceil^2}{2} + \left(\frac{1-\hat{h}}{2}\right)(1 - 2\lceil h \rceil) + T_h\left(\hat{h}\right)$$

with $\hat{h} := h \mod 1$ $(h \notin \mathbb{N}), \hat{h} := 1$ $(h \in \mathbb{N}), \hat{h} := 0$ $(h = 0)$,
where $T_h(x)$ is a de Rham-type function.

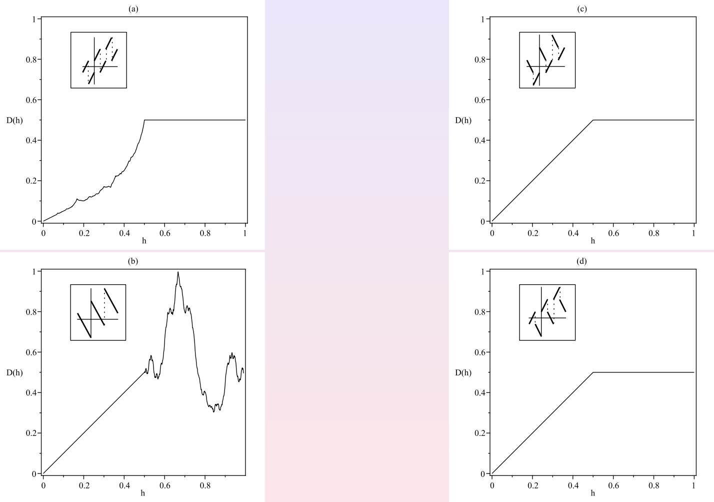


on large scales we recover the random walk solution



on small scales $D(h)$ is again *partially* a fractal function

. . . and further strange diffusion coefficients

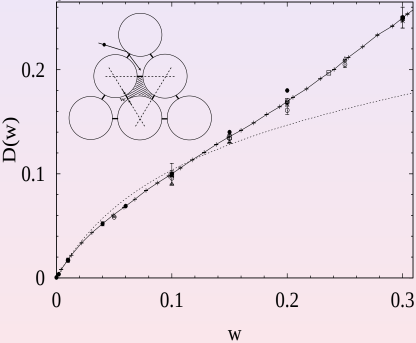


Knight, RK, Nonlinearity (2011)

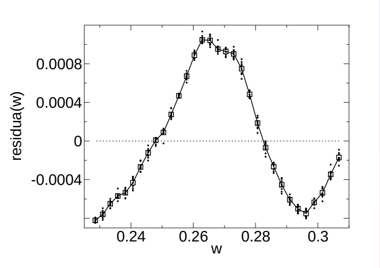
Irregular diffusion coefficients in the Lorentz gas

diffusion coefficient *D* vs. minimal spacing *w* between two nearby scatterers (Klages, Dellago, 2000)

computer simulation results:



residua for large *w*:



dots (left): random walk approx. (Machta, Zwanzig, 1983)

$\Rightarrow$ periodic Lorentz gas exhibits 'irregular' diffusion coefficient

open question: degree of smoothness?

Summary

- How does the **diffusion coefficient** of a chaotic map depend on **size** and **position** of a hole?

two surprising results:

1. **size:** contrary to intuition, a smaller hole may yield a larger diffusion coefficient
2. **position:** violation of simple random walk approximation for the diffusion coefficient if the hole converges to a *periodic orbit*

- intimate relation between **hole-dependence of escape** and **fractality of parameter-dependent diffusion coefficients**

References

- **hole-dependent diffusion:**

  G.Knight, O.Georgiou, C.P.Dettmann, R.Klages,
  preprint arXiv:1112.3922 (2011)

- **fractal diffusion coefficients:**