A simple non-chaotic map generating subdiffusive, diffusive and superdiffusive dynamics

L. Salari[1] L. Rondoni[1,2] C. Giberti[3] R. Klages[4,5]

[1]Dipartimento di Scienze Matematiche, Politecnico di Torino
[2]GraphenePoliTO Lab, Politecnico di Torino and INFN Sezione di Torino
[3]Dipt. di Scienze e Metodi dell Ingegneria, Universita di Modena e Reggio E.
[4]Max Planck Institute for the Physics of Complex Systems, Dresden
[5]Queen Mary University of London, School of Mathematical Sciences

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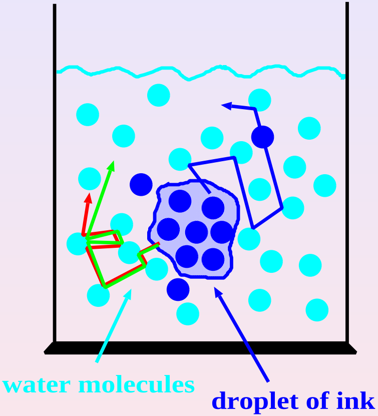
Outline

1. **Motivation:** chaos, diffusion and polygonal billiards
2. **Model:** mimick diffusion in polygonal billiards by a simple non-chaotic map
3. **Results:** non-trivial diffusive properties matching to different known stochastic processes

Microscopic chaos in a glass of water?



- dispersion of a droplet of ink by diffusion
- chaotic collisions between billiard balls
- **chaotic hypothesis:**

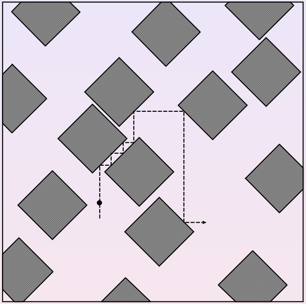
microscopic chaos
⇓
macroscopic diffusion

Gallavotti, Cohen (1995)

P.Gaspard et al. (1998): experiment on small colloidal particle in water; diffusion due to microscopic chaos based on positive *pattern entropy per unit time* $h(\epsilon, \tau) \leq h_{KS} = \sum_{\lambda_i > 0} \lambda_i$

The random wind tree model

counterexample:



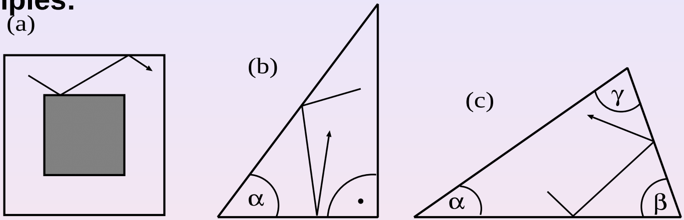
Ehrenfest, Ehrenfest (1959)

no positive Lyapunov exponent, hence non-chaotic dynamics

Dettmann et al. (1999): generates trajectories and $h(\epsilon, \tau)$ *indistinguishable from the colloidal particle dynamics*

Polygonal billiards

examples:



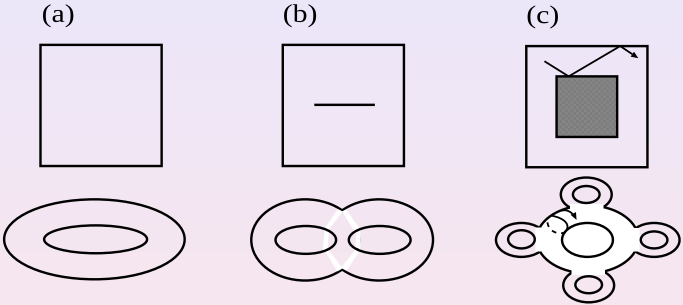
Artuso et al. (1997,2000); Casati et al. (1999)

rational billiards: all angles are rational multiples of π
irrational billiards: otherwise

non-trivial ergodic properties: rational billiards are not ergodic; phase space splits into invariant manifolds wrt initial angle of trajectory (e.g., Gutkin, 1996)

Pseudointegrability

joining all identical edges yields compact invariant surfaces:

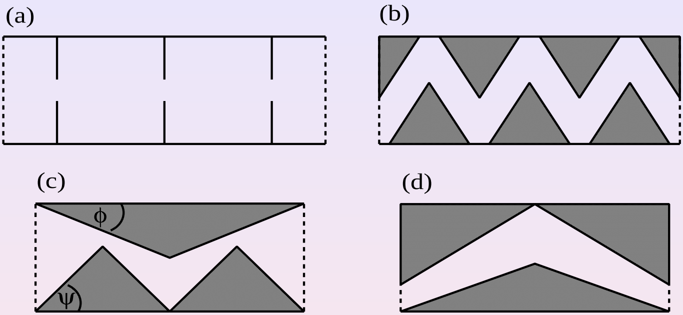


genus $g = 1$: billiard is *integrable*

$g > 1$: pseudointegrable (Richens, Berry, 1981); $\exists$ isolated saddles resembling hyperbolic fixed points imposing a 'chaotic character' onto the flow

asymptotic growth of displacement of two trajectories $\Delta(t) \sim t$

Diffusion in polygonal billiard channels



Zwanzig (1983), Zaslavsky et al. (2001), Li et al. (2002)

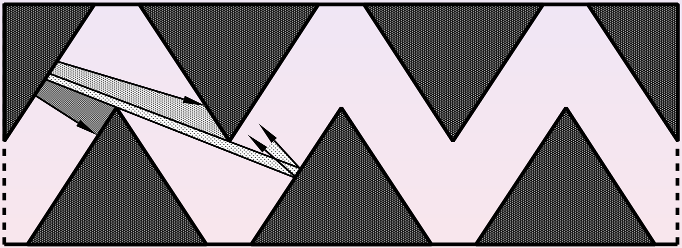
mean square displacement $< x^2 > := \int dx\, x^2 \rho(x,t) \sim t^\gamma$
from simulations: sub- ($\gamma < 1$), super- ($\gamma > 1$) or normal ($\gamma = 1$) diffusion depending on parameters; partially conflicting results

Alonso et al. (2002), Jepps et al. (2006), Sanders et al. (2006)

Particle dispersion in polygonal billiards

simple picture:

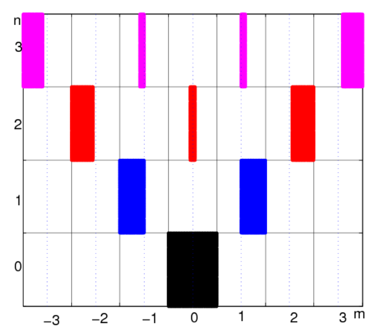
diffusion in these channels may be crucially determined by how scatterers slice a beam



this should be captured by interval exchange transformations (Hannay, McCraw, 1990)

The slicer map I: basic idea

a simple one-dimensional *spatially dependent* interval exchange transformation:



zero Lyaponuv exponent: different points neither converge nor diverge from each other in time; slicer points are of Lebesgue measure zero; hence non-chaotic dynamics

The slicer map II: definition of the model

- consider a **chain of intervals** $\widehat{M} := M \times \mathbb{Z}$, $M := [0,1]$ with point $\widehat{X} = (x,m)$ in $\widehat{M}$, where $\widehat{M}_m := M \times \{m\}$ is the m-th cell of $\widehat{M}$

- subdivide each $\widehat{M}_m$ in 4 subintervals, separated by 3 points called **slicers**: $\{1/2\} \times \{m\}$, $\{\ell_m\} \times \{m\}$, $\{1-\ell_m\} \times \{m\}$, where $0 < \ell_m < 1/2$ for every $m \in \mathbb{Z}$ with

$$\ell_m(\alpha) = \frac{1}{\left(|m|+2^{1/\alpha}\right)^{\alpha}}, \ \alpha > 0$$

- **slicer map:** $S : \widehat{M} \to \widehat{M}$, $\widehat{X}_{n+1} = S(\widehat{X}_n)$, $n \in \mathbb{N}$ with

$$S(x,m) = \begin{cases} (x, m-1) & \text{if } 0 \le x < \ell_m \text{ or } \frac{1}{2} < x \le 1-\ell_m, \\ (x, m+1) & \text{if } \ell_m \le x \le \frac{1}{2} \text{ or } 1-\ell_m < x \le 1. \end{cases}$$

Spreading under slicer action

choose initial density as

$$\hat{\rho}_0\left(\widehat{X}\right) = \begin{cases} 1, & \text{if } \widehat{X} \in \widehat{M}_0 \\ 0, & \text{otherwise} \end{cases}$$

which is *chopped* under the action of S to

$$\hat{\rho}_n(\widehat{X}) = \begin{cases} 1 & \text{if } \widehat{X} \in S^n \widehat{M}_0 \\ 0 & \text{otherwise} \end{cases}$$

the sets $\widehat{R}_j := S^n \widehat{M_0} \cap \widehat{M}_j$, $j = -n, \ldots, n$, constitute the total phase space volume occupied at time n in cell $\widehat{M}_j$

the (Lebesgue) measure $A_j = \hat{\mu}(\widehat{R}_j)$ of $\widehat{R}_j$ equals the probability of being in cell j at time n yielding the coarse grained distribution

$$\rho_n^G(j) = \begin{cases} A_j & \text{if } j \in \{-n, \ldots, n\}, \\ 0 & \text{otherwise} \end{cases}$$

Diffusion in the slicer map

based on ρ_n^G define the mean square displacement $\langle \Delta \hat{X}_n^2 \rangle := \sum_{j=-n}^{n} A_j j^2$ with distance j travelled by a point in $\widehat{M}_j$ at time n

Proposition

Given $\alpha \in [0,2]$ and a uniform initial distribution in $\widehat{M}_0$, we have

1. $\alpha = 0$: *ballistic motion with* $\langle x_n^2 \rangle \sim n^2$
2. $0 < \alpha < 1$: *superdiffusion with MSD* $\langle x_n^2 \rangle \sim n^{2-\alpha}$
3. $\alpha = 1$: *normal diffusion with linear MSD* $\langle x_n^2 \rangle \sim n$
   note: *non-chaotic normal diffusion with non-Gaussian density*
4. $1 < \alpha < 2$: *subdiffusion with MSD* $\langle x_n^2 \rangle \sim n^{2-\alpha}$
   note: *subdiffusion with ballistic peaks*
5. $\alpha = 2$: *logarithmic subdiffusion with MSD* $\langle x_n^2 \rangle \sim \log n$
6. $\alpha > 2$: *localisation in the MSD with* $\langle x_n^2 \rangle \sim const.$

The higher order moments in the slicer

Theorem

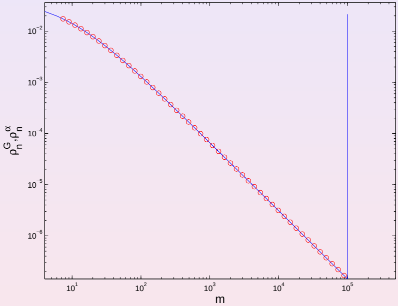
For $\alpha \in (0, 2]$ the moments $\langle \Delta\hat{X}_n^p \rangle = \sum_{j=-n}^{n} A_j j^p$ with $p > 2$ even and initial condition uniform in $\widehat{M}_0$ have the asymptotic behavior

$$\langle \Delta\hat{X}_n^p \rangle \sim n^{p-\alpha}$$

while the odd moments ($p = 1, 3, ...$) vanish.

Example: $\alpha = 1/3$

we have $\langle \Delta \widehat{X}_n^p \rangle \sim n^{p-1/3}$ and especially $\langle \Delta \widehat{X}_n^2 \rangle \sim n^{5/3}$:
superdiffusion; plot of analytic $\rho_n^G(m)$ (continuous line):



cp. with asymptotics: $\rho_n^\alpha(m) = \begin{cases} \dfrac{C_\alpha}{(m+2^{1/\alpha})^{\alpha+1}}, & m < n \\ 0, & m > n \end{cases}$

with normalisation C_α; note peak in the traveling area

Matching to stochastic dynamics?

- one-dimensional stochastic Lévy Lorentz gas:

point particle moves ballistically between static point scatterers on a line from which it is transmitted / reflected with probability $1/2$

distance r between two scatterers is a random variable iid from the Lévy distribution

$$\lambda(r) \equiv \beta r_0^\beta \frac{1}{r^{\beta+1}} \; , \; r \in [r_0, +\infty) \; , \; \beta > 0$$

with cutoff r_0

$\rightarrow$ model exhibits only *superdiffusion*

$\rightarrow$ *all moments scale with the slicer moments* for $\alpha \in (0, 1]$ (piecewise linearly depending on parameters)

Matching to stochastic dynamics?

- Lévy walk modeled by CTRW theory:

$\rightarrow$ *moments* calculated to $\sim t^{p+1-\beta}$ for $p > \beta$, $1 < \beta < 2$:
match to slicer *superdiffusion* with $\beta = 1 + \alpha$

$\rightarrow$ but conceptually a totally *different process*

- correlated Gaussian stochastic processes:

modeled by a generalized Langevin equation with a power law memory kernel

$\rightarrow$ formal analogy in the *subdiffusive* regime

$\rightarrow$ but Gaussian distribution and a *conceptual mismatch*

Summary

- **central theme:**
  diffusion generated by *non-chaotic dynamics*
- **main result:**
  slicer model generates 6 different types of diffusive dynamics under parameter variation covering the whole spectrum of diffusion
- this result might help to explain a **controversy about different stochastic models for diffusion in polygonal billiards**: sensitive dependence of diffusion on parameters matching to different stochastic processes

References

- slicer:
L.Salari, L.Rondoni, C.Giberti, RK, Chaos **25**, 073113 (2015)

- review about polygonal billiards: Section 17.4 in
R.Klages, *Microscopic Chaos, Fractals and Transport in Nonequilibrium Statistical Mechanics* (World Scientific, 2007)